A TEST FOR HOLOMORPHY IN THE UNIT BALL OF $\mathbf{C}^n$

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ABSTRACT. A new test for holomorphy in the unit ball of $\mathbf{C}^n$ in the spirit of Morera's theorem is given.

The purpose of this note is to give a new criterion for holomorphy in the unit ball $\mathbf{B}^n$ of $\mathbf{C}^n$ ($n \geq 1$) in the spirit of the generalization of the classical Morera theorem given by Zalcman in the case of the complex plane [10]. We denote by $\mathfrak{M}$ the group of biholomorphic mappings of $\mathbf{B}^n$. For $n = 1$ this is just the classical Moebius group; we refer to [8] for a description in the general case. $\mathbf{B}^n$ is a rank-one symmetric space and, as such, it has an $\mathfrak{M}$ -invariant metric, the Poincaré metric and a corresponding $\mathfrak{M}$ -invariant measure $d\mu = \rho(z) d\lambda$, where $d\lambda$ is the Lebesgue measure and $\rho(z) = (1 - |z|^2)^{-2n}$. In order to integrate over hypersurfaces we need differential forms generalizing dz from $n = 1$ to higher dimensions; these are the $(n, n - 1)$ forms

$$\omega_k = dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_k} \wedge \cdots \wedge d\bar{z}_n, \quad 1 \leq k \leq n.$$

Let Ω be an open, nonempty, relatively compact subset of $\mathbf{B}^n$, and assume that $\partial\Omega$ is a piecewise C^1 -manifold, so that the Stokes theorem can be applied. If f is smooth in $\mathbf{B}^n$, we have

$$(1) \quad \int_{\partial\Omega} f \omega_k = \int_{\Omega} df \wedge \omega_k = c_{n,k} \int_{\Omega} \frac{\partial f}{\partial \bar{z}_k} d\lambda,$$

where $c_{n,k} = (2i)^n (-1)^{\frac{1}{2}n(n-1)+k-1}$.

THEOREM. *Assume that, in addition to the above, $\mathbf{B}^n \setminus \Omega$ is connected and that $\partial\Omega$ is not a real analytic manifold. Then a continuous function f in $\mathbf{B}^n$ is holomorphic if and only if for every $\sigma \in \mathfrak{M}$*

$$(2) \quad \int_{\sigma(\partial\Omega)} f \omega_k = 0, \quad k = 1, 2, \dots, n.$$

PROOF. The necessity of condition (2) follows from (1) above. To prove sufficiency note first that by mollifying the original function using a radial approximate identity we can assume that $f \in C^\infty(\mathbf{B}^n)$. If f is not holomorphic then, for some k , $\partial f / \partial \bar{z}_k$ is

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not identically zero. Applying (1) to the nonzero function $g = \rho^{-1}(\partial f / \partial \bar{z}_k)$, we obtain

$$(3) \quad \int_{\sigma(\Omega)} g \, d\mu = 0 \quad \forall \sigma \in \mathfrak{N}.$$

This is just the Pompeiu problem for Ω , and Proposition 2 from [3] implies that there is a positive eigenvalue α and a corresponding eigenfunction u for the overdetermined Neumann problem

$$(4) \quad \begin{aligned} \Delta u + \alpha u &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0, \quad u = 1 \quad \text{on } \partial\Omega \end{aligned}$$

where Δ denotes the Laplace-Beltrami operator in $\mathbf{B}^n$. The work of Nirenberg et al. (cf. [9]) on such overdetermined problems implies that $\partial\Omega$ is a real analytic manifold. Hence we have arrived to a contradiction and the proof is finished.

Although the result is most striking in the context of the unit ball, an analogous theorem holds when $\mathbf{B}^n$ is replaced by $\mathbf{C}^n$ and $\mathfrak{N}$ by the group $M(2n)$ of Euclidean motions of $\mathbf{R}^{2n}$. In fact, the same proof applies verbatim, the result from [3] being replaced by the earlier [6, 4].

The theorem leaves unanswered the case where $\partial\Omega$ is real analytic. Now, it is known that if Ω is a ball in $\mathbf{B}^n$ (same remark applies to $\mathbf{C}^n$), there are many nonholomorphic functions f satisfying (2). In fact, it is enough to exhibit $g \neq 0$ satisfying the condition (3) and solve the equation $\partial f / \partial \bar{z}_1 = \rho g$. For the Pompeiu problem (3) and Ω a ball in $\mathbf{B}^n$ one can explicitly give many such $g \neq 0$ (see e.g. [4, p. 610]) or one can use the fact that for any Ω with connected exterior, the Pompeiu problem (3) and the overdetermined eigenvalue problem (4) are equivalent (see [3]). The relation between the existence of such eigenvalues and the geometry of Ω is discussed in [2, 5]. We also suggest [11] for an overview of these problems.

Agranovskii [1] has obtained results similar to the above theorem without any condition on the real analyticity of $\partial\Omega$, however his result applies only to functions f for which $\rho^{-1}(\partial f / \partial \bar{z}_k) \in L^2(\mathbf{B}^n, d\mu)$ for $k = 1, \dots, n$ (he also shows that when $n = 1$, $f \in L^2(\mathbf{B}^1, d\lambda)$ suffices). This is a severe limitation; for instance, the corresponding result in $\mathbf{C}^n$ would only test the constants. The same paper contains a different criterion for holomorphy in symmetric spaces which is obtained by Banach algebras techniques (see also [7]).

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