A REMARK ON THE HARDY-LITTLEWOOD INEQUALITY

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ABSTRACT. We show that the inequality of the Littlewood conjecture holds precisely for compact abelian groups whose connected component has finite index.

The Littlewood conjecture was settled independently by McGehee, Pigno, and Smith [2] and by Konyagin [3]. Both the statement of "Hardy's inequality" [2, Theorem 2] and its proof require a total group ordering on the character group $\hat{G}$. Such an ordering exists precisely when $\hat{G}$ is torsion-free, i.e. when G is connected.

At the other extreme, Hewitt and Zuckerman noticed in [1] that if $\hat{G}$ has an infinite torsion subgroup, then for arbitrarily large $N \geq 1$ there are sums of N characters $f = \sum_{k=1}^N \chi_k$ with $\|f\|_1 = 1$, so that the conjecture certainly fails to extend to arbitrary compact abelian G . The following theorem completes the picture.

THEOREM. *Let G be any compact abelian group, the dual $\hat{G}$ of which has exactly n elements of finite order. Equivalently, the identity component G_0 of G has finite index n . Then there is a constant $K > 0$ such that*

$$(*) \quad \left\| \sum_{k=1}^N c_k \gamma_k \right\|_1 \geq K \log N$$

holds whenever $\gamma_1, \dots, \gamma_N$ are distinct characters, and $c_1, \dots, c_N$ are complex numbers with modulus ≥ 1 .

PROOF. The following proof is simpler than that first submitted, and is due independently to the referee and D. I. Cartwright. Let $f = \sum_{k=1}^N c_k \gamma_k$ be given. We may replace $\hat{G}$ by the subgroup generated by $\gamma_1, \dots, \gamma_N$ together with the n torsion elements of $\hat{G}$. This amounts to replacing G by a compact quotient, and f by a trigonometric polynomial of exactly the same form on that quotient. This done, we see that $\hat{G}$ is finitely generated, and so $\hat{G} = \hat{A} \times \mathbf{Z}^m$, and $G = A \times G_0$ as a topological group, where A and its dual $\hat{A}$ are finite groups of order n .

There is a $\gamma \in \hat{A}$ such that

$$\text{Card}(\text{Supp } \hat{f} \cap (\{\gamma\} \times \mathbf{Z}^m)) \geq N/n.$$

Let μ denote Haar measure on G_0 , normalized so that $\|\mu\| = 1$. Put $g = \mu * (\bar{\gamma}f)$. Then g is a trigonometric polynomial of the form $\sum_{j=1}^p c'_j \gamma'_j$ where $|c'_j| \geq 1$ ($1 \leq j \leq p$) and $p \geq N/n$. Set $g_0 = g|_{G_0}$. Since $\text{supp } \hat{g} \subseteq \{0\} \times \mathbf{Z}^m$, $\|g\|_{L^1(G)} = n\|g_0\|_{L^1(G)}$. By [2] applied to G_0 there is a constant $K_0 > 0$ depending only on G_0 such that

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$\|g_0\|_{L^1(G)} \geq K_0 \log p$. We then have

$$\begin{aligned} \|f\|_{L^1(G)} &\geq \|g\|_{L^1(G)} = n\|g_0\|_{L^1(G)} \\ &= \|g_0\|_{L^1(G_0)} \geq K_0 \log p \geq K_0 \log(N/n). \end{aligned}$$

Hence a $K > 0$ exists which depends only on G and for which $\|f\|_1 \geq K \log N$.

REFERENCES

1. Edwin Hewitt and Herbert S. Zuckerman, *On a theorem of P. J. Cohen and H. Davenport*, Proc. Amer. Math. Soc. **14** (1963), 847–855.
2. O. Carruth McGeehee, Louis Pigno and Brent Smith, *Hardy's inequality and the L^1 norm of exponential sums*, Ann. of Math. (2) **113** (1981), 613–618.
3. S. V. Konyagin, *On the Littlewood problem*, Izv. Akad. Nauk USSR Ser. Fiz.-Mat. Nauk **45** (1981), 243–265.

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