ON THE PROBLEM OF MODIFIED MOMENTS

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ABSTRACT. The problem of modified moments is studied. Let $(P_n(x))_{n=0}^{\infty}$ be an orthogonal polynomial sequence. Given a sequence $(d_n)_{n=0}^{\infty}$ of real numbers, does there exist a bounded nondecreasing function with infinitely many points of increase such that for every $n \in \mathbb{N}_0$, $d_n = \int_{-\infty}^{\infty} P_n(x) d\mu(x)$? Is there any information about the support of μ ? A necessary and sufficient condition for the existence of such a function μ is given in terms of the positivity of certain determinants. For certain $(P_n(x))_{n=0}^{\infty}$ a description of the support of μ is established.

Let $(P_n(x))_{n=0}^{\infty}$ be an orthogonal polynomial sequence. A sequence $(d_n)_{n=0}^{\infty}$ of real numbers is called a sequence of modified moments with respect to $(P_n(x))_{n=0}^{\infty}$ if there exists a bounded nondecreasing function μ with infinitely many points of increase such that

$$d_n = \int_{-\infty}^{\infty} P_n(x) d\mu(x), \quad n = 0, 1, 2, \dots$$

Modified moments are studied, e.g., in numerical analysis; see [2, p. 47].

Write the linearization of the product of any two orthogonal polynomials $P_m(x)$ and $P_n(x)$:

$$(L) \quad P_m(x)P_n(x) = \sum_{k=0}^{2\min(m,n)} g(m, n, m+n-k)P_{m+n-k}(x).$$

Given $(d_n)_{n=0}^{\infty}$ denote

$$d_{nm} = d_{mn} = \sum_{k=0}^{2\min(m,n)} g(m, n, m+n-k)d_{m+n-k}.$$

The classical theory of moments yields the following solution of the problem of modified moments.

THEOREM 1. *Let $(P_n(x))_{n=0}^{\infty}$ be an orthogonal polynomial sequence. Given a real-valued sequence $(d_n)_{n=0}^{\infty}$, the following are equivalent:*

(a) $d_n = \int_{-\infty}^{\infty} P_n(x) d\mu(x)$, where μ is a nondecreasing function with infinitely many points of increase.

(b) $\det((d_{ij})_{0 \leq i, j \leq n}) > 0$ for $n = 0, 1, 2, \dots$

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PROOF. Assume at first that (b) is valid. Define a functional ψ on the space of all polynomials by

$$\psi(p) = \sum_{n=0}^N a_n d_n,$$

provided $p(x) = \sum_{n=0}^N a_n P_n(x)$. By (b) we have $\psi(p^2) > 0$ for each $p \neq 0$. Now [6, Chapter VI, problem 44] and Theorem 1.1 of [7] imply there exists a bounded nondecreasing function μ such that $\int_{-\infty}^{\infty} P_n(x) d\mu(x) = d_n$. The function μ has infinitely many points of increase. In fact, assume $\text{supp } \mu = \{x_1, \dots, x_n\}$. Consider a polynomial $p \neq 0$ such that $p(x_i) = 0$ for $i = 1, \dots, n$. Then

$$p(x) = \sum_{n=0}^N a_n P_n(x)$$

and

$$\sum_{n,m=0}^N a_n a_m d_{nm} = \int_{-\infty}^{\infty} p^2(x) d\mu(x) = 0,$$

a contradiction to (b).

Having more information about $P_n(x)$ one might ask for a stronger result. Assume the orthogonal polynomials $P_n(x)$ are random walk polynomials, that is

$$P_0(x) = 1, \quad a_0 P_1(x) = x - b_0,$$

$$a_n P_{n+1}(x) = P_1(x) P_n(x) - b_n P_n(x) - c_n P_{n-1}(x),$$

where $a_0 > 0$, $b_0 \in \mathbf{R}$ with $a_0 + b_0 = 1$ and $a_n > 0$, $b_n \geq 0$, $c_n > 0$ with $a_n + b_n + c_n = 1$, $n \in \mathbf{N}$. Moreover, assume each linearization coefficient $g(m, n, m + n - k)$ in (L) is nonnegative. Then we shall say $(P_n(x))_{n=0}^{\infty}$ satisfies property (P). In [5] we have established that there is an intimate relation between hypergroup structures on $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and orthogonal polynomial sequences having the positivity property (P). Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$, where $\alpha \geq \beta > -1$, $\alpha + \beta + 1 \geq 0$, associated Legendre polynomials $L_n^{\nu}(x)$, where $\nu \geq 0$, or continuous q -ultraspherical polynomials $C_n(x; \beta | q)$ where $-1 < \beta < 1$, $0 < q < 1$ belong to this class. Many other examples can be found in [5]. Given $(P_n(x))_{n=0}^{\infty}$ satisfying property (P), the character space of the corresponding hypergroup $\mathbf{N}_0$ can be identified with

$$D_S = \{x \in \mathbf{R}: (P_n(x))_{n=0}^{\infty} \text{ is bounded}\}.$$

One knows that D_S is compact and $\text{supp } \pi \subseteq D_S \subseteq [1 - 2a_0, 1]$ where π is the orthogonalization function. Each character on $\mathbf{N}_0$ is given by $\alpha_x: \mathbf{N}_0 \rightarrow \mathbf{R}$, $\alpha_x(n) = P_n(x)$, where $x \in D_S$. Compare [5].

THEOREM 2. *Let $(P_n(x))_{n=0}^{\infty}$ be an orthogonal polynomial sequence satisfying property (P). Given a bounded real-valued sequence $(d_n)_{n=0}^{\infty}$ the following are equivalent:*

(a) $d_n = \int_{D_S} P_n(x) d\mu(x)$, where μ is a nondecreasing function with infinitely many points of increase in D_S .

(b) $\det((d_{ij})_{0 \leq i, j \leq n}) > 0$ for $n = 0, 1, 2, \dots$

PROOF. Using Theorem 12.3B of [3] condition (b) yields a nondecreasing function μ with all points of increase in D_S such that $d_n = \int_{D_S} P_n(x) d\mu(x)$. The remaining assertions of Theorem 2 follow as in the proof of Theorem 1.

REMARKS. (a) I want to thank the referee for his useful comments. (b) In [4] a theory of moment problems for T -systems is developed a concept encompassing orthogonal polynomial sets. Further, in [1, No. 12] C. Berg considered a stronger version of positive definiteness with respect to certain polynomials, including the ultraspherical polynomials.

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