ON A THEOREM OF P. L. ULJANOV

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ABSTRACT. It is shown that if $c(n) = o(1)$, $|n| \rightarrow \infty$, and $\sum_{|n| < \infty} |\Delta^m c(n)| < \infty$, for some integer $m \geq 1$, then the series $\sum_{|n| < \infty} c(n)e^{int}$ converges to some $f \in L^p(\mathbf{T})$ for any $0 < p < 1/m$.

As an application of A -integrals, P. L. Uljanov [1] obtained an interesting result concerning the integrability of $|f|^p$ and $|\tilde{f}|^p$, for any $0 < p < 1$, where

$$(1) \quad f(x) = \sum_{k=1}^{\infty} a_k \cos kx,$$

$$(2) \quad \tilde{f}(x) = \sum_{k=1}^{\infty} a_k \sin kx$$

and $\{a_n\}$ is a null-sequence of bounded variation.

THEOREM (P. L. ULJANOV [1]). *Let $\{a_n\}$ be a null-sequence of bounded variation. Then for any $0 < p < 1$, $|f|^p$ and $|\tilde{f}|^p$ are integrable, where f and $\tilde{f}$ are defined by (1) and (2).*

In this paper I shall prove a version of Uljanov's theorem and extend it to the series

$$(3) \quad \sum_{|n| < \infty} c(n)e^{int}, \quad t \in \mathbf{T} = \mathbf{R}/2\pi\mathbf{Z},$$

where $\{c(n)\}_{|n| < \infty}$ is a null-sequence of complex numbers such that for some integer $m \geq 1$

$$(4) \quad \sum_{|n| < \infty} |\Delta^m c(n)| < \infty,$$

where $\Delta^m c(n) = \Delta^{m-1} c(n) - \Delta^{m-1} c(n+1)$. J. W. Garrett, C. S. Rees and Č. V. Stanojević [2] considered (4) in the case of real sequences. They also gave an example in [2] to show that (4) is an effective generalization of sequences of bounded variation.

In this paper I shall establish the following result.

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THEOREM. Let $\{c(n)\}_{|n| < \infty}$ be a null-sequence of complex numbers. If, for some integer $m \geq 1$, (4) holds, then the point-wise limit f of

$$(5) \quad S_n(t) = \sum_{|j| \leq n} c(j) e^{ijt}$$

exists in $\mathbf{T} - \{0\}$ and for any $0 < p < 1/m$, $|f|^p$ and $|\bar{f}|^p$ are integrable.

PROOF. Let $m \geq 1$ and let $M < N$. Define

$$A_{M,N+1}^{(k)} = A_{M,N+1}^{(k)}(t) = [\Delta^k c(M)] e^{iMt} - [\Delta^k c(N+1)] e^{i(N+1)t}$$

and $w = w(t) = 1 - e^{-it}$. Then the following identity can be established:

$$w^m \sum_{j=M+m}^{N+m} c(j) e^{ijt} = \sum_{j=M}^N [\Delta^m c(j)] e^{ijt} - \sum_{k=0}^{m-1} w^k A_{M+k, N+k+1}^{(m-1-k)}$$

for any sequence $\{c(n)\}_{|n| < \infty}$ of complex numbers. Consequently, setting $M = -n$, $N = n$, for $t \neq 0$ we have

$$(6) \quad S_n(t) = \frac{1}{w^m} \sum_{|j| \leq n} [\Delta^m c(j)] e^{ijt} - \frac{1}{w^m} \sum_{k=0}^{m-1} w^k A_{-n+k, n+k+1}^{(m-1-k)} \\ + \sum_{j=-n}^{-n+m-1} c(j) e^{ijt} - \sum_{j=n+1}^{n+m} c(j) e^{ijt}.$$

Since (4) holds, the first term on the right-hand side of (6) converges point-wise in $\mathbf{T} - \{0\}$. The rest of the terms tend to zero because $c(n) = o(1)$, $|n| \rightarrow \infty$. Hence the limit of $S_n(t)$, $n \rightarrow \infty$, exists in $\mathbf{T} - \{0\}$ and we denote it by f .

It remains to prove that for any $0 < p < 1/m$, $|f| \in L^p(\mathbf{T})$. Since $S_n(t)$ is a trigonometric polynomial it suffices to show that $S_n(t)$ converges to f in $L^p(\mathbf{T})$ -metric for any $0 < p < 1/m$. For $t \neq 0$ consider

$$f(t) - S_n(t) = \frac{1}{w^m} \sum_{|n| < \infty} [\Delta^m c(n)] e^{int} \\ - \frac{1}{w^m} \sum_{|j| \leq n} [\Delta^m c(j)] e^{ijt} + \frac{1}{w^m} \sum_{k=0}^{m-1} w^k A_{-n+k, n+k+1}^{(m-1-k)} \\ - \sum_{j=-n}^{-n+m-1} c(j) e^{ijt} + \sum_{j=n+1}^{n+m} c(j) e^{ijt}.$$

Hence we have

$$f(t) - S_n(t) = \frac{1}{w^m} \sum_{|j| \geq n+1} [\Delta^m c(j)] e^{ijt} + \frac{1}{w^m} \sum_{k=0}^{m-1} w^k A_{-n+k, n+k+1}^{(m-1-k)} \\ - \sum_{j=-n}^{-n+m-1} c(j) e^{ijt} + \sum_{j=n+1}^{n+m} c(j) e^{ijt}.$$

For $0 < p < 1/m$ we obtain

$$\begin{aligned} |f(t) - S_n(t)|^p &\leq \frac{1}{|w(t)|^{m \cdot p}} \left(\sum_{|j| \geq n+1} |\Delta^m c(j)| \right)^p \\ &\quad + \frac{2^{m-1}}{|w(t)|^{mp}} \left(\sum_{k=0}^{m-1} (|\Delta^{(m-1-k)} c(-n+k)| + |\Delta^{(m-1-k)} c(n+k+1)|) \right)^p \\ &\quad + \left(\sum_{j=-n}^{-n+m-1} |c(j)| \right)^p + \left(\sum_{j=n+1}^{n+m} |c(j)| \right)^p. \end{aligned}$$

Since for $mp < 1$

$$\int_{\mathbf{T}} \frac{dt}{|w(t)|^{mp}} < K,$$

where K is an absolute constant, we have

$$\begin{aligned} \int_{\mathbf{T}} |f(t) - S_n(t)|^p dt &\leq K \left(\sum_{|j| \geq n+1} |\Delta^m c(j)| \right)^p \\ &\quad + 2^{m-1} K \left(\sum_{k=0}^{m-1} (|\Delta^{(m-1-k)} c(-n+k)| + |\Delta^{(m-1-k)} c(n+k+1)|) \right)^p \\ &\quad + m^p \max_{-n \leq j \leq -n+m-1} |c(j)|^p + m^p \max_{n+1 \leq j \leq n+m} |c(j)|^p. \end{aligned}$$

The second sum on the right-hand side of the last inequality is a finite sum of $o(1)$ terms, as $n \rightarrow \infty$, and the last two terms are trivially $o(1)$. Hence

$$\left(\int_{\mathbf{T}} |f(t) - S_n(t)|^p dt \right)^{1/p} = O \left(\sum_{|j| \geq n+1} |\Delta^m c(j)| \right), \quad n \rightarrow \infty.$$

Finally, because of (4),

$$\left(\int_{\mathbf{T}} |f(t) - S_n(t)|^p dt \right)^{1/p} = o(1), \quad n \rightarrow \infty.$$

Thus $|f| \in L^p(\mathbf{T})$ for any $0 < p < 1/m$. That concludes the proof of the Theorem.

In a similar manner we can obtain that $|\tilde{f}| \in L^p(\mathbf{T})$ for any $0 < p < 1/m$. For $m = 1$ we have the theorem of P. L. Uljanov.

The case $m = 2$ is related to some subtle questions about H^p -spaces and it will be studied in a subsequent paper.

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