DILATIONS OF V -BOUNDED STOCHASTIC PROCESSES INDEXED BY A LOCALLY COMPACT GROUP

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ABSTRACT. It is proved that a stochastic process (i.e., a Hilbert space valued function) indexed by a locally compact group is V -bounded (i.e., weakly harmonizable in an appropriate sense) if, and only if, it can be expressed as an orthogonal projection of a process whose covariance function R satisfies $R(s, t) = \rho(t^{-1}s) + \rho(st^{-1})$ for some continuous positive-definite function ρ . The result generalizes a well-known theorem due to H. Niemi, and depends on the noncommutative Grothendieck type inequality of G. Pisier.

Let G be a locally compact (not necessarily Abelian) group and H a (complex) Hilbert space. In analogy with the case $G = \mathbf{R}$ and $H = L_0^2(\Omega, \mathcal{A}, P) = \{f \in L^2 \mid \int f dP = 0\}$ for a probability space $(\Omega, \mathcal{A}, P)$, we call any function $\phi: G \rightarrow H$ a (stochastic) process. A process $\phi: G \rightarrow H$ is called V -bounded, if it is weakly continuous and the set $\{\int \phi d\mu \mid \mu \in L^1(G), \|\mu\|' \leq 1\}$ is bounded in H , where $\|\mu\|'$ denotes the least upper bound of $\|\pi(\mu)\|$ when π ranges over the continuous unitary representations of G . As shown in [6], this is equivalent to a property that might be called *weak harmonizability* as in the commutative case, i.e., that there is a bounded linear map $\Phi: C^*(G) \rightarrow H$ whose Fourier transform $\hat{\Phi}$ equals ϕ . Here $\hat{\Phi}$ is defined by the formula $\hat{\Phi}(s) = \Phi^{**}(\omega(s))$ where $\Phi^{**}: C^*(G)^{**} \rightarrow H$ is the bitranspose of Φ and $\omega: G \rightarrow C^*(G)^{**} \subset L(H_\omega)$ is the representation of G corresponding to the universal representation of the group C^* -algebra $C^*(G)$ of G . For a detailed discussion of these matters we refer to [6].

In [4], H. Niemi proved that in the case $G = \mathbf{R}$ a process is V -bounded if, and only if, it is an orthogonal projection of a stationary process. For subsequent developments of the theme one may, e.g., consult [1] and its bibliography. In this note we extend Niemi's result to the noncommutative setting. Our treatment is also in part influenced by [1 and 2]. Let us begin with a general lemma, which explains the role of 2-majorizability, inherent in the subject.

LEMMA 1. *Let A be an involutive algebra, $\Phi: A \rightarrow H$ a linear map and $f_1, f_2: A \rightarrow \mathbf{C}$ positive linear forms (i.e., $f_j(x^*x) \geq 0$ for all $x \in A$, $j = 1, 2$). The following two conditions are equivalent.*

- (i) $\|\Phi x\|^2 \leq f_1(x^*x) + f_2(xx^*)$ for all $x \in A$.
- (ii) *There is a Hilbert space K such that for some linear map $\Psi: A \rightarrow K$ and some isometric linear map $V: H \rightarrow K$, $\Phi = V^*\Psi$ and $(\Psi x \mid \Psi y) = f_1(y^*x) + f_2(xy^*)$ whenever $x, y \in A$.*

PROOF. First assume (i). Then (ii) can be proved along the lines of [2, pp. 256–257]. Alternatively, denote $h(x, y) = f_1(y^*x) + f_2(xy^*)$ and $N = \{x \in A \mid h(x, x) = 0\}$.

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Since h is a positive Hermitian form, N is a linear subspace of A , and A/N can be completed to a Hilbert space W with inner product $(x + N|y + N) = h(x, y)$. Now (i) implies the existence of a linear contraction $\Phi_0: W \rightarrow H$ with $\Phi_0(x + N) = \Phi x$. Then as in [1, p. 274] an idea going back to [3] yields a Hilbert space K along with isometric linear maps $\Psi_0: W \rightarrow K$ and $V: H \rightarrow K$ such that $\Phi_0 = V^* \Psi_0$. Defining $\Psi x = \Psi_0(x + N)$ we get the Ψ required for (ii); since Ψ_0 preserves inner products, $(\Psi x|\Psi y) = h(x, y)$. It is, conversely, clear that (ii) implies (i), since $\|\Phi\| \leq \|\Psi\|$.

As usual, $B(G)$ will stand for the vector space spanned by the continuous positive-definite functions on G . As in [6, p. 359], we denote by $T: B(G) \rightarrow C^*(G)$ the bijection satisfying $\rho(s) = \langle \omega(s), T\rho \rangle$ for $s \in G$.

LEMMA 2. *A process $\phi: G \rightarrow H$ is V -bounded, if there are two functions $\rho_1, \rho_2 \in B(G)$ such that*

$$(1) \quad (\phi(s)|\phi(t)) = \rho_1(t^{-1}s) + \rho_2(st^{-1}) \quad \text{for all } s, t \in G.$$

PROOF. From (1) it follows that ϕ is norm continuous. Denote $f_i = T\rho_i \in C^*(G)^*$, $i = 1, 2$. If $s_j \in G$, $\lambda_j \in \mathbb{C}$ and $\mu = \sum_{j=1}^n \lambda_j \delta_{s_j}$ is the corresponding linear combination of point measures, we have

$$\begin{aligned} \left\| \sum_{j=1}^n \lambda_j \phi(s_j) \right\|^2 &= \sum_{j=1}^n \sum_{k=1}^n \lambda_j \bar{\lambda}_k [f_1(\omega(s_k^{-1})\omega(s_j)) + f_2(\omega(s_j)\omega(s_k^{-1}))] \\ &= f_1(\omega(\mu)^* \omega(\mu)) + f_2(\omega(\mu)\omega(\mu)^*) \leq (\|f_1\| + \|f_2\|)(\|\mu\|)^2 \end{aligned}$$

(see e.g. [6, pp. 358–359]). Thus ϕ is V -bounded (see Remark 6.3 in [6, p. 379]).

We are now ready to prove our main result, which generalizes Theorem 11 in [4, p. 124].

THEOREM. *For a stochastic process $\phi: G \rightarrow H$ the following two conditions are equivalent.*

- (i) ϕ is V -bounded.
- (ii) *There is a Hilbert space K and a process $\psi: G \rightarrow K$ such that, for some isometric linear map $V: H \rightarrow K$ and for some continuous positive-definite function $\rho: G \rightarrow \mathbb{C}$, $\phi = V^* \circ \psi$ and $(\psi(s)|\psi(t)) = \rho(t^{-1}s) + \rho(st^{-1})$ whenever $s, t \in G$.*

PROOF. If ψ is as in (ii), it is V -bounded by Lemma 2, and so $\phi = V^* \circ \psi$ is V -bounded, too. Now assume (i). Remark 6.3 in [6, p. 379] shows that ϕ is the Fourier transform of some bounded linear map $\Phi: C^*(G) \rightarrow H$. There is a positive linear functional $f: C^*(G) \rightarrow \mathbb{C}$ such that $\|\Phi x\|^2 \leq f(x^*x + xx^*)$ for all $x \in C^*(G)$ (see [5, p. 400]), and so Lemma 1 yields a Hilbert space K and a linear map $\Psi: C^*(G) \rightarrow K$ such that $(\Psi x|\Psi y) = f(y^*x) + f(xy^*)$ for all $x, y \in C^*(G)$ and that for some isometric linear map $V: H \rightarrow K$ we have $\Phi = V^* \Psi$. Since $\|\Psi x\|^2 \leq 2\|f\|\|x\|^2$, Ψ is bounded, and defining ψ as its Fourier transform (see [6, p. 360]) we obviously have $\phi = V^* \circ \psi$.

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