THE F. AND M. RIESZ THEOREM AND SINGULAR MEASURES

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Dedicated to Professor Haruo Sunouchi on his sixtieth birthday.

ABSTRACT. Some generalizations of the F. and M. Riesz theorem are given.

Let $M(\mathbf{R}^N)$ denote the Banach space of all complex Borel measures on the N -dimensional Euclidean space $\mathbf{R}^N$ (cf. [9]). We define the Fourier transform of a measure $\mu \in M(\mathbf{R}^N)$ by

$$\hat{\mu}(t) = \int e^{-itx} d\mu(x) \quad (t \in \mathbf{R}^N),$$

where tx denotes the inner product of $t, x \in \mathbf{R}^N$. The F. and M. Riesz theorem states that if a measure $\mu \in M(\mathbf{R})$ is concentrated on a half-open interval of finite length l and if $\hat{\mu}(2n\pi/l) = 0$ for all negative integers n , then μ is absolutely continuous. For an elementary proof of this classical result, we refer to the nice paper by R. Doss [4]. It is an easy consequence of the above-mentioned theorem that if $\mu \in M(\mathbf{R})$ and $\hat{\mu}(t) = 0$ for all $t < 0$, then μ is absolutely continuous (see, e.g., 8.2.7 of [9]).

In the present note, we shall give some generalizations of the last result. For a subset E of $\mathbf{R}^N$, let $\text{sp}(E)$ denote the linear span of all e^{-itx} with $t \in E$. Given $\mu \in M(\mathbf{R}^N)$, define

$$\|\mu\|_E = \sup \left\{ \left| \int P d\mu \right| : P \in \text{sp}(E), \|P\|_\infty < 1 \right\}.$$

THEOREM 1. *Let (M_k) , (N_k) and (ϵ_k) be three sequences of real positive numbers such that*

$$\lim_{k \rightarrow \infty} N_k = \infty \quad \text{and} \quad \lim_{k \rightarrow \infty} \epsilon_k = 0.$$

For each $k \geq 1$, let

$$E_k = \{mM_k + n\epsilon_k : m, n \in \mathbf{N}, n < N_k/\epsilon_k\}.$$

Then we have $\|\mu\| = \lim_{k \rightarrow \infty} \|\mu\|_{E_k}$ for each singular measure $\mu \in M(\mathbf{R})$.

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COROLLARY 1.1. Let (M_k) , (N_k) and (ε_k) be as in the above theorem. If $\mu \in M(\mathbf{R})$ and if

$$\sum_{m=1}^{\infty} |\hat{\mu}(mM_k + n\varepsilon_k)|^2 < \infty \quad (1 \leq n < N_k/\varepsilon_k)$$

holds for every $k \geq 1$, then μ is absolutely continuous.

COROLLARY 1.2. Let a and b be two real positive numbers which are rationally independent, and let (M_n) be a sequence of natural numbers. If $\mu \in M(\mathbf{R})$ satisfies

$$\sum_{m=1}^{\infty} |\hat{\mu}((mM_n - r)a + nb)|^2 < \infty \quad (1 \leq r < nb/a)$$

for every $n \geq 1$, then μ is absolutely continuous.

The last corollary is an improvement of F. Forelli's result [5].

PROOF OF THEOREM 1. Let a singular measure $\mu \in M(\mathbf{R})$ be given. For each $k \geq 1$, put

$$S_k = \{mM_k; m \in \mathbf{N}\} \quad \text{and} \quad T_k = \{n\varepsilon_k; n \in \mathbf{N} \text{ \& } 1 \leq n < N_k/\varepsilon_k\}.$$

In order to prove the desired result, it will suffice to show that

$$(a) \quad \inf\{\|(f-1)\mu\|: f \in \text{sp}(S_k), \|f\|_{\infty} < 1\} = 0$$

for each $k \geq 1$, and that

$$(b) \quad \|\mu\| = \lim_{k \rightarrow \infty} \|\mu\|_{T_k}.$$

Without loss of generality, we may and do assume that $\|\mu\| = 1$ and that μ has compact support K with Lebesgue measure zero.

Let $\varepsilon > 0$ and $k \geq 1$ be given. Then notice that

$$[K + (2\pi/M_k)\mathbf{Z}] \cap [0, 2\pi/M_k]$$

is a compact subset of $[0, 2\pi/M_k]$ having Lebesgue measure zero. It follows from the F. and M. Riesz theorem (or equivalently, Rudin's Theorem [8]) that there exists an $f \in \text{sp}(S_k)$ such that $\|f\|_{\infty} < 1$ and $|f-1| < \varepsilon$ on K (cf. the proof of Theorem 2 *infra*). Since $\varepsilon > 0$ was arbitrary, this establishes (a).

Now we shall confirm (b). Choose and fix a real positive number a such that $K \subset (-A, A)$, where $A = a^{-1}$. Since μ is a singular measure concentrated on $(-A, A)$, the F. and M. Riesz theorem yields a trigonometric polynomial h of the form

$$(1) \quad h(x) = \sum_{n=1}^N c_n e^{-inax} \quad (x \in \mathbf{R})$$

such that

$$(2) \quad \|h\|_{\infty} < 1 \quad \text{and} \quad \left| \int h d\mu \right| > 1 - \varepsilon.$$

Choose $\eta > 0$ so small that $\eta A \sum_{n=1}^N |C_n| < \varepsilon$.

Since $\varepsilon_k \rightarrow 0$ and $N_k \rightarrow \infty$ as $k \rightarrow \infty$ by the hypotheses, every $k > k(\eta)$ satisfies the conditions

$$(3) \quad |q_k \varepsilon_k - a| < \eta/N \quad \text{and} \quad Na + \eta < N_k$$

for some $q_k \in \mathbf{N}$. Define a trigonometric polynomial g_k by setting

$$(4) \quad g_k(x) = \sum_{n=1}^N C_n \exp(-inq_k \epsilon_k x).$$

Notice that (3) implies

$$nq_k \epsilon_k < N(a + \eta/N) < N_k \quad (1 \leq n \leq N).$$

Hence (4) shows that g_k is in $\text{sp}(T_k)$. Moreover, $\|g_k\|_\infty = \|h\|_\infty < 1$ by (1), (2) and (4). If $|x| < A$, then

$$|h(x) - g_k(x)| \leq \sum_{n=1}^N |C_n| \cdot |n(a - q_k \epsilon_k)| \cdot |x| \leq \eta A \sum_{n=1}^N |C_n| < \epsilon$$

by (1), (4) and (3). Since μ is concentrated on $(-A, A)$, it follows from (2) and the last inequalities that

$$\left| \int g_k d\mu \right| > \left| \int h d\mu \right| - \epsilon > 1 - 2\epsilon.$$

(Recall that $\|\mu\| = 1$.) This confirms (b) and the proof is complete.

PROOF OF COROLLARY 1.1. Let $\mu \in M(\mathbf{R})$ be as in the hypotheses of the present corollary. Replacing each M_k by an appropriate integral multiple thereof, we may and do assume that

$$(5) \quad M_k > k + N_k \epsilon_k$$

and

$$(6) \quad \sum \left\{ |\hat{\mu}(t)|^2 : t \in E_k \right\} < k^{-1}$$

for all $k \geq 1$.

Given $k \geq 1$, let P_k be an arbitrary member of $\text{sp}(E_k)$ with $\|P_k\|_\infty < 1$. Thus we can write

$$P_k(x) = \sum_{m=1}^M C_m \exp(-it_m x),$$

where the $C_m = C_{m,k}$ and the $t_m = t_{m,k}$ are complex numbers and different elements of E_k , respectively. Then we have

$$(7) \quad \left| \int P_k d\mu \right| = \left| \sum_{m=1}^M C_m \hat{\mu}(t_m) \right| \leq \left(\sum_{m=1}^M |C_m|^2 \right)^{1/2} \left(\sum_{m=1}^M |\hat{\mu}(t_m)|^2 \right)^{1/2} \leq \|P_k\|_\infty / k < k^{-1}$$

by Schwarz' inequality and (6). Notice that $t \in E_k$ implies $|t| > k$ by (5) and the definition of E_k . It follows that $\lim_{k \rightarrow \infty} |\int P_k d\mu_a| = 0$, where $\mu_a[\mu_s]$ denotes the absolutely continuous [singular] part of μ (cf. [3]). This, combined with (7), yields $\lim_{k \rightarrow \infty} |\int P_k d\mu_s| = 0$. It follows from Theorem 1 that $\mu_s = 0$, as desired.

PROOF OF COROLLARY 1.2. For each $k \geq 1$, choose two natural numbers p_k and q_k so that $0 < q_k b - p_k a < 1/k$; such a choice is possible since a and b are rationally

independent. Define $\varepsilon_k = q_k b - p_k a$, $N_k = k$, and $M'_k = M_1 M_2 \cdots M_u$, where $u = u(k) \in \mathbf{N}$ is greater than $k q_k / \varepsilon_k$. If $m, n \in \mathbf{N}$ and $n < k / \varepsilon_k$, then we have, writing $p = p_k$ and $q = q_k$,

$$m M'_k a + n \varepsilon_k = [m M'_k - np]a + nqb = [m(M'_k / M_{nq})M_{nq} - np]a + nqb,$$

and M'_k / M_{nq} is an integer. Thus Corollary 1.2 follows from Corollary 1.1.

THEOREM 2. Suppose $N \geq 2$ and $\mu \in M(\mathbf{R}^N)$ satisfies the following two conditions:

- (i) $\hat{\mu}(t) = 0$ for all $t = (t_j) \in \mathbf{R}^N$ with $t_1 \geq 0$, and
- (ii) for each $t_1 < 0$, $\hat{\mu}(t_1, \cdot)$ is the Fourier transform of some $f_{t_1} \in L^1(\mathbf{R}^{N-1})$.

Then μ is absolutely continuous.

COROLLARY 2.1 (BOCHNER'S THEOREM [1]). Let $\mu \in M(\mathbf{T}^N)$ be given. Suppose that there exist $\varepsilon > 0$ and a nonzero vector $a = (a_j) \in \mathbf{R}^N$ such that $\hat{\mu}$ vanishes on the set

$$\{n \in \mathbf{Z}^N: a_1 n_1 + \cdots + a_N n_N < \varepsilon \|n\|\}.$$

Then μ is absolutely continuous.

PROOF OF THEOREM 2. Let (m_k) be any sequence of real positive numbers with $\lim_{k \rightarrow \infty} m_k = \infty$ (e.g., $m_k = k$). For each $k \geq 1$, define $E_k = m_k E$ and $F_k = m_k F$, where

$$E = \mathbf{N} \times \mathbf{Z}^{N-1} \quad \text{and} \quad F = (\{0\} \times \mathbf{Z}^{N-1}) \setminus \{0\}.$$

Then we claim that the hypotheses of Theorem 2 imply

$$(8) \quad \lim_{k \rightarrow \infty} \|e_t \mu\|_{E_k} = 0 = \lim_{k \rightarrow \infty} \|e_t \mu\|_{F_k}$$

for each $t \in \mathbf{R}^N$. (Recall that $e_t(x) = e^{-itx}$ for $t, x \in \mathbf{R}^N$.) Indeed, the first equality in (8) is evident by (i); the second equality also follows from (i) if $t_1 \geq 0$. So assume $u = t_1 < 0$ and write $t = (u, v)$ with $v \in \mathbf{R}^{N-1}$. Regarding F_k as a subset of $\mathbf{R}^{N-1}$ in the obvious way, we then have $\|e_t \mu\|_{F_k} = \|e_v f_u\|_{F_k}$, where $f_u \in L^1(\mathbf{R}^{N-1})$ is as in (ii). Since $m_k \rightarrow \infty$, it is obvious that the sets F_k are eventually disjoint from any (fixed) compact subset of $\mathbf{R}^{N-1}$. We therefore have the second equality in (8) even in the case $t_1 < 0$ (cf. [3]).

Now we claim that (8) implies

$$(9) \quad \lim_{k \rightarrow \infty} \|e_t \mu\|_{H_k} = 0 \quad (t \in \mathbf{R}^N),$$

where $H_k = m_k(\mathbf{Z}_+ \times \mathbf{Z}^{N-1}) \setminus \{0\}$ for each $k \geq 1$. To confirm this, we may assume $t = 0$ by replacing μ by $e_t \mu$. Fix $k \geq 1$ and choose any $P \in \text{sp}(H_k)$ with $\|P\|_\infty < 1$. Since H_k is the union of the disjoint sets E_k and F_k , P can be uniquely written as $P = Q + R$, where $Q \in \text{sp}(E_k)$ and $R \in \text{sp}(F_k)$. Consider the measure $\lambda = [(2\pi)^{-1} m_k] \lambda' \times \delta$, where $\lambda' \in M(\mathbf{R})$ is the Lebesgue measure restricted to the interval $(0, 2\pi/m_k)$ and δ is the unit point-measure at $0 \in \mathbf{R}^{N-1}$. One checks that $\hat{\lambda} = 1$ on F_k and $\hat{\lambda} = 0$ on E_k . It follows that $R = \lambda * P$, so $\|R\|_\infty \leq \|P\|_\infty < 1$ and $\|Q\|_\infty < 2$. (This result is a special case of Proposition 4.1 of [10].) We therefore have

$$\left| \int P d\mu \right| \leq \left| \int Q d\mu \right| + \left| \int R d\mu \right| \leq 2 \|\mu\|_{E_k} + \|\mu\|_{F_k}.$$

Since P is an arbitrary element of $\text{sp}(H_k)$ with $\|P\|_\infty < 1$, the last inequalities yield $\|\mu\|_{H_k} \leq 2\|\mu\|_{E_k} + \|\mu\|_{F_k}$ for all $k \geq 1$. Thus (8) implies (9).

In order to complete the proof, it will suffice to verify that every measure satisfying (9) is absolutely continuous. It is obvious that (9) holds for each absolutely continuous measure. Therefore we may and do assume that μ is a singular measure satisfying (9); of course, then we must prove that $\mu = 0$.

Given $\varepsilon > 0$, choose and fix a compact subset K of $\mathbf{R}^N$ with Lebesgue measure zero such that $\|\mu|_K\| > \|\mu\| - \varepsilon$. For each $k \geq 1$, the set $K + m_k^{-1}2\pi\mathbf{Z}^N$ is a closed set having Lebesgue measure zero and "period" $m_k^{-1}2\pi\mathbf{Z}^N$. It follows that there exists an $f_k \in \text{sp}(H_k)$ such that $\|f_k\|_\infty < 1$ and $|f_k - 1| < \varepsilon$ on K . Otherwise, the Hahn-Banach theorem combined with the Riesz representation theorem would yield a singular measure $\nu \in M(\mathbf{T}^N)$ such that $\hat{\nu}(0) = 1$ and $\|\nu\|_H < 1$, where $H = (\mathbf{Z}_+ \times \mathbf{Z}^{N-1}) \setminus \{0\}$. Then we would obtain a measure $\rho \in M(\mathbf{T}^N)$ such that $\hat{\rho} = \hat{\nu}$ on H and $\|\rho\| < 1$ (again by the above-mentioned two theorems). Hence $\hat{\nu} - \hat{\rho} = 0$ on H ; but

$$|\hat{\nu}_s(0) - \hat{\rho}_s(0)| = |\hat{\nu}(0) - \hat{\rho}(0)| > 1 - \|\rho\| > 0,$$

which contradicts Theorem 8.2.3 of [9]. (Notice that $\mathbf{Z}^N \setminus H$ is contained in a semigroup in $\mathbf{Z}^N$ which induces an order in $\mathbf{Z}^N$.)

Now, if f_k is as in the above paragraph, then

$$\left| \int f_k d\mu - \hat{\mu}(0) \right| \leq \|(f_k - 1)\mu\| \leq \varepsilon \|\mu|_K\| + 2(\|\mu\| - \|\mu|_K\|) < (\|\mu\| + 2)\varepsilon,$$

so

$$|\hat{\mu}(0)| < \left| \int f_k d\mu \right| + (\|\mu\| + 2)\varepsilon \leq \|\mu\|_{H_k} + (\|\mu\| + 2)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the last inequalities combined with (9) give us $\hat{\mu}(0) = 0$. Applying this result to $e_t\mu$, we obtain $\hat{\mu}(t) = 0$ for all $t \in \mathbf{R}^N$, so $\mu = 0$, as desired.

PROOF OF COROLLARY 2.1. Let $\tau \in M(\mathbf{R}^N)$ be any probability measure such that $\hat{\tau}$ is supported by the open unit ball in $\mathbf{R}^N$. Given $\mu \in M(\mathbf{T}^N)$, define a function $f = f_\mu \in C(\mathbf{R}^N)$ by setting

$$(10) \quad f(t) = \sum_n \hat{\mu}(n) \hat{\tau}(t - n) \quad (t \in \mathbf{R}^N).$$

By Lemma 1 of [7], we have $f = \hat{\nu}$ for some $\nu \in M(\mathbf{R}^N)$ and μ is absolutely continuous if and only if ν is absolutely continuous.

Now let $a \in \mathbf{R}^N$, $\varepsilon > 0$, and $\mu \in M(\mathbf{T}^N)$ be as in the hypotheses of the present corollary, and let $\nu \in M(\mathbf{R}^N)$ be defined as in the above paragraph. We may and do assume that a is a unit vector and that $0 < \varepsilon < 1$. Let θ be a rotation of $\mathbf{R}^N$ such that $\theta(-1, 0, \dots, 0) = a$. A simple calculation shows that $\text{supp}(\hat{\nu} \circ \theta)$ is contained in the cone $w + \{y \in \mathbf{R}^N: -y_1 \geq \varepsilon\|y\|\}$, where $w = (-(1 - \varepsilon^2)^{-1/2}, 0, \dots, 0)$. Thus, upon defining $\rho \in M(\mathbf{R}^N)$ by $\hat{\rho}(t) = \hat{\nu}(\theta(t + w))$ for $t \in \mathbf{R}^N$, we see that $\hat{\rho} = 0$ on the half-space $\mathbf{R}_+ \times \mathbf{R}^{N-1}$ and that for each $t_1 < 0$, $\hat{\rho}(t_1, \cdot)$ has compact support (in $\mathbf{R}^{N-1}$). We therefore infer from Theorem 2 that ρ is absolutely continuous; hence so are ν and μ . This completes the proof.

REMARK. Let (m_k) be a sequence of natural numbers, and let E be the union of all $m_k\mathbf{N} + k$ with $k \geq 1$. Then $\mathbf{Z} \setminus E$ is a Riesz set, i.e., if $\mu \in M(\mathbf{T})$ and $\hat{\mu} = 0$ on E , then μ is absolutely continuous. This fact is obvious from the proof of Theorem 1. Notice that, by requiring that $m_k \rightarrow \infty$ very rapidly, we can make $\delta(E)$ arbitrarily small, where $\delta(E)$ denotes the natural density of E :

$$\delta(E) = \limsup_{n \rightarrow \infty} n^{-1} \text{Card}[E \cap \{1, 2, \dots, n\}].$$

It is not difficult to show that there exists *no* set $E \subset \mathbf{N}$, with $\delta(E) = 0$, for which $\mathbf{Z} \setminus E$ is a Riesz set. It is possible, however, to prove that given a σ -compact subset K of $\mathbf{T}$ with Lebesgue measure zero, there *does* exist $E \subset \mathbf{N}$, with $\delta(E) = 0$, such that $\|\mu\| = \|\mu\|_E$ for all $\mu \in M(K)$. In particular, in the Rudin-Carleson theorem, we may replace $\mathbf{N}$ by a subset of $\mathbf{N}$ with natural density 0. We omit the details.

The results of this paper are generalizations of a note which I wrote to Professor I. Glicksberg in 1977. I would like to thank him for encouraging me to publish them. The referee points out that our methods are similar to those in [11].

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