A NOTE ON THE ERGODIC THEOREM

TAKESHI YOSHIMOTO

ABSTRACT. Let M be a positive regular shift-invariant method of summability and let T be a one-to-one transformation which maps X onto X and which is $\mathfrak{B}$ -bimeasurable, i.e., $A \in \mathfrak{B}$ if and only if $TA \in \mathfrak{B}$, where $(X, \mathfrak{B})$ is a measurable space. Then it is proved that if for a finite measure μ on $\mathfrak{B}$ the sequence $\{\mu(T^m A)\}$ is M -summable for each $A \in \mathfrak{B}$, then for any real $r \geq 1$ the sequence $\{f \circ T^m\}$ is (C, r) -summable μ -almost everywhere for every bounded $\mathfrak{B}$ -measurable function f defined on X . The result includes the Blum-Hanson theorem.

Let X be a nonempty set, $\mathfrak{B}$ a σ -algebra of subsets of X , and Y a reflexive Banach space with norm denoted by $\|\cdot\|$. The symbol $L(X: Y)$ will stand for the class of all bounded strongly $\mathfrak{B}$ -measurable Y -valued functions defined on X . Let $M = (m_{ij})$ ($i, j = 0, 1, 2, \dots$) be an infinite matrix satisfying:

- (i) $m_{ij} \geq 0$ for all i and j ,
- (ii) $\sum_{j=0}^{\infty} m_{ij} \leq C < \infty$ uniformly in i ,
- (iii) $\lim_{i \rightarrow \infty} m_{ij} = 0$ for all j , and
- (iv) $\lim_{i \rightarrow \infty} \sum_{j=0}^{\infty} m_{ij} = 1$.

Such a matrix M defines a positive regular method of summability and a good deal is known about such methods. The method M is called shift-invariant provided for every sequence $\{u_n\}$ of real numbers for which $\lim_{i \rightarrow \infty} \sum_{j=0}^{\infty} m_{ij} u_j = u$ exists and is finite we have $\lim_{i \rightarrow \infty} \sum_{j=0}^{\infty} m_{ij} u_{j+1} = u$. Fix a real $r \geq 1$ arbitrarily and put $A_0^{(r)} = 1$ and $A_n^{(r)} = (r+1)(r+2) \cdots (r+n)/n!$ for $n = 1, 2, 3, \dots$. The (C, r) -method, i.e., $m_{ij}^{(r)} = A_{i-j}^{(r-1)}/A_i^{(r)}$ for $j = 0, 1, \dots, i$ and $m_{ij}^{(r)} = 0$ for $j = i+1, i+2, \dots$, is an example of a positive regular shift-invariant method of summability. A sequence $\{u_n\}$ is called M -summable if $\lim_{i \rightarrow \infty} \sum_{j=0}^{\infty} m_{ij} u_j$ exists. We now prove the following (C, r) -generalization of the Blum-Hanson theorem [1].

THEOREM 1. *Let μ be a finite measure defined on $\mathfrak{B}$, T a one-to-one $\mathfrak{B}$ -bimeasurable transformation on X , and M a positive regular shift-invariant method of summability. If $\{\mu(T^m A)\}$ is M -summable for each $A \in \mathfrak{B}$ then the sequence $\{f \circ T^m\}$ is (C, r) -summable (strongly in Y) μ -almost everywhere for every $f \in L(X: Y)$, i.e., there exists a function $f^* \in L(X: Y)$ such that*

$$(a) \quad f^*(x) = \lim_{n \rightarrow \infty} \sum_{m=0}^n (A_{n-m}^{(r)}/A_n^{(r)}) f(T^m x),$$

Received by the editors May 5, 1983.

1980 *Mathematics Subject Classification*. Primary 47A35; Secondary 28C05.

Key words and phrases. Regular summation method, M -summable, (C, r) -summable, Blum-Hanson theorem.

©1984 American Mathematical Society
0002-9939/84 \$1.00 + \$.25 per page

$$(b) \quad f^*(Tx) = f^*(x)$$

(strongly in Y) for μ -almost all $x \in X$.

In what follows, the symbol U_T will be used for denoting the induced operator of T , i.e., $U_T f = f \circ T$. We note that the class of finite measures for which our theorem holds includes the class of finite T -invariant measures. In particular, if one takes $r = 1$ and Y to be the linear space of real or complex numbers, then our result becomes the Blum-Hanson theorem. The operator-theoretical treatment of (C, r) -summability, including the σ -finite case of invariant measures, is given in [4].

PROOF OF THEOREM 1. Let $\lambda(A) = \lim_{i \rightarrow \infty} \lambda_i(A)$ for each $A \in \mathfrak{B}$, where $\lambda_i(A) = \sum_{j=0}^{\infty} m_{ij} \mu(T^j A)$ ($i = 0, 1, 2, \dots$). Then λ is a finite measure on $\mathfrak{B}$ which is invariant under T . Indeed this follows from the shift-invariance of M and the fact that if λ_n is a sequence of finite signed measures such that $\lambda(A) = \lim_{n \rightarrow \infty} \lambda_n(A)$ exists and is finite for every $A \in \mathfrak{B}$, then λ is a signed measure (see, for example, Halmos [3, p. 170]). Clearly, μ is absolutely continuous with respect to λ , i.e., $\lambda(A) = 0$, $A \in \mathfrak{B}$ implies $\mu(A) = 0$. Since $L_2(X, \mathfrak{B}, \lambda, Y)$ is reflexive, it follows that by the Yosida-Kakutani mean ergodic theorem, the functions f of the form $f = g + (I - U_T)h$, $g = U_T g$, $g \in L_2(X, \mathfrak{B}, \lambda, Y)$, $h \in L_{\infty}(X, \mathfrak{B}, \lambda, Y)$, are dense in $L_2(X, \mathfrak{B}, \lambda, Y)$. For such a function f , one gets

$$\begin{aligned} \sum_{m=0}^n \frac{A_{n-m}^{(r-1)}}{A_n^{(r)}} U_T^m f(x) &= \left(\sum_{m=0}^n \frac{A_{n-m}^{(r-1)}}{A_n^{(r)}} \right) g(x) + \frac{A_n^{(r-1)}}{A_n^{(r)}} h(x) - \frac{A_0^{(r-1)}}{A_n^{(r)}} U_T^{n+1} h(x) \\ &\quad + \sum_{m=1}^n \left[\frac{A_{n-m}^{(r-1)} - A_{n-m+1}^{(r-1)}}{A_n^{(r)}} \right] U_T^m h(x). \end{aligned}$$

Note that $\sum_{m=0}^n A_{n-m}^{(r-1)} = A_n^{(r)}$ and $\lim_{n \rightarrow \infty} A_n^{(r)} = \infty$. It follows that

$$\lim_{n \rightarrow \infty} \sum_{m=0}^n \frac{A_{n-m}^{(r-1)}}{A_n^{(r)}} g(x) = g(x), \quad \lim_{n \rightarrow \infty} \frac{A_n^{(r-1)}}{A_n^{(r)}} h(x) = 0$$

and

$$\lim_{n \rightarrow \infty} \frac{A_0^{(r-1)}}{A_n^{(r)}} U_T^{n+1} h(x) = 0$$

strongly in Y for λ -almost all $x \in X$. Furthermore, for any integer $p > 1$,

$$\begin{aligned} \left\| \sum_{m=1}^n \left[\frac{A_{n-m}^{(r-1)} - A_{n-m+1}^{(r-1)}}{A_n^{(r)}} \right] U_T^m h(x) \right\| &\leq \|h\|_{\infty} \sum_{m=1}^{\infty} \left| \frac{A_{n-m}^{(r-1)} - A_{n-m+1}^{(r-1)}}{A_n^{(r)}} \right| \\ &\leq \|h\|_{\infty} \left(\sum_{m=1}^{p-1} \left| \frac{A_{n-m}^{(r-1)} - A_{n-m+1}^{(r-1)}}{A_n^{(r)}} \right| + \sum_{m=p}^{\infty} \left| \frac{A_{n-m}^{(r-1)} - A_{n-m+1}^{(r-1)}}{A_n^{(r)}} \right| \right). \end{aligned}$$

Now the Cesàro matrix $(m_{ij}^{(r)})$ satisfies the uniformity condition in the sense of Cohen:

$$\lim_{p \rightarrow \infty} \sum_{j=p}^{\infty} |m_{i,j+1}^{(r)} - m_{ij}^{(r)}| = 0$$

uniformly in i (see Cohen [2]). So, given $\varepsilon > 0$, there is a number $n(\varepsilon)$ such that

$$\left\| \sum_{m=0}^n \left[\frac{A_{n-m}^{(r-1)} - A_{n-m+1}^{(r-1)}}{A_n^{(r)}} \right] U_T^m h(x) \right\| \leq 2 \|h\|_\infty \varepsilon, \quad n > n(\varepsilon).$$

Thus the limit, $\lim_{n \rightarrow \infty} \sum_{m=0}^n (A_{n-m}^{(r-1)} / A_n^{(r)}) U_T^m f$, exists (in Y) λ -almost everywhere for every f in a set dense in $L_2(X, \mathfrak{B}, \lambda, Y)$. However,

$$\sup_{n \geq 0} \left\| \sum_{m=0}^n \frac{A_{n-m}^{(r-1)}}{A_n^{(r)}} U_T^m f \right\| \leq r \sup_{n \geq 0} \left\| \frac{1}{n+1} \sum_{m=0}^n U_T^m f \right\| < \infty$$

λ -almost everywhere for any $f \in L_2(X, \mathfrak{B}, \lambda, Y)$. Therefore by the Banach convergence theorem, $\lim_{n \rightarrow \infty} \sum_{m=0}^n (A_{n-m}^{(r-1)} / A_n^{(r)}) U_T^m f$ exists (in Y) λ -almost everywhere for every $f \in L_2(X, \mathfrak{B}, \lambda, Y)$. Since $L(X: Y) \subset L_2(X, \mathfrak{B}, \lambda, Y)$ by the finiteness of λ , Theorem 1(a) follows at once from this by the absolute continuity of μ with respect to λ . To see Theorem 1(b) it suffices to note that given $\varepsilon > 0$ there is a number $n(\varepsilon)$ such that, for $f \in L(X: Y)$,

$$\begin{aligned} & \left\| \sum_{m=0}^n \frac{A_{n-m}^{(r-1)}}{A_n^{(r)}} U_T^m f - \sum_{m=0}^n \frac{A_{n-m}^{(r-1)}}{A_n^{(r)}} U_T^{m+1} f \right\| \\ & \leq \left\| \frac{A_n^{(r-1)}}{A_n^{(r)}} f \right\| + \left\| \frac{A_0^{(r-1)}}{A_n^{(r)}} U_T^{n+1} f \right\| + \left\| \sum_{m=1}^n \left[\frac{A_{n-m}^{(r-1)} - A_{n-m+1}^{(r-1)}}{A_n^{(r)}} \right] U_T^m f \right\| \\ & < 4 \|f\|_\infty \varepsilon, \quad n > n(\varepsilon). \end{aligned}$$

Hence Theorem 1 has been proved.

The following corollaries are immediate from Theorem 1 using the Lebesgue bounded convergence theorem.

COROLLARY 1 (CF. [1, COROLLARY 2]). *On the hypothesis of Theorem 1, if the sequence $\{\mu(T^m A)\}$ is M -summable for each $A \in \mathfrak{B}$ then it is also (C, r) -summable.*

COROLLARY 2 (CF. [1, COROLLARY 1]). *Let T be a one-to-one $\mathfrak{B}$ -bimeasurable transformation on X and let μ be a finite measure on $\mathfrak{B}$. Then the following statements are equivalent:*

- (1) *the sequence $\{\mu(T^m A)\}$ is $(C, 1)$ -summable for each $A \in \mathfrak{B}$,*
- (2) *the sequence $\{\mu(T^m A)\}$ is (C, r) -summable for each $A \in \mathfrak{B}$,*
- (3) *the sequence $\{U_T^m f\}$ is $(C, 1)$ -summable μ -almost everywhere for each $f \in L(X: Y)$,*
- (4) *the sequence $\{U_T^m f\}$ is (C, r) -summable μ -almost everywhere for each $f \in L(X: Y)$.*

More generally, we consider a regular summation method $A = (a_{nm}: n, m = 0, 1, 2, \dots)$. Such methods are characterized by the familiar Silverman-Toeplitz conditions (ii)–(iv) mentioned above. Modifying the argument used in the proof of

Theorem 1 (cf. [4]) we have

THEOREM 2. *Let $A = (a_{nm})$ be a regular summation method satisfying the uniformity condition in the sense of Cohen. Let T be a $\mathfrak{B}$ -measurable point transformation on X and μ a σ -finite measure on $\mathfrak{B}$ such that U_T is contracting on $L_1(X, \mathfrak{B}, \mu, Y)$ as well as on $L_\infty(X, \mathfrak{B}, \mu, Y)$. Assume there is a positive constant C such that for any $f \in L_p(X, \mathfrak{B}, \mu, Y)$, $1 \leq p < \infty$,*

$$\sup_n \left\| \sum_{m=0}^{\infty} a_{nm} U_T^m f \right\| \leq C \sup_n \left\| \frac{1}{n+1} \sum_{m=0}^n U_T^m f \right\|$$

μ -almost everywhere. Then the sequence $\{U_T^m f\}$ is A -summable (strongly in Y) μ -almost everywhere for each $f \in L_p(X, \mathfrak{B}, \mu, Y)$, $1 \leq p < \infty$.

REFERENCES

1. J. R. Blum and D. L. Hanson, *A note on the ergodic theorem*, Proc. Amer. Math. Soc. **16** (1965), 413–414.
2. L. W. Cohen, *On the mean ergodic theorem*, Ann. of Math. (2) **41** (1940), 505–509.
3. P. R. Halmos, *Measure theory*, Van Nostrand, New York, 1950.
4. T. Yoshimoto, *Generalized ergodic inequalities and ergodic theorems*, Probability Theory and Mathematical Statistics (ed. by K. Ito and J. V. Prokhorov): Proceedings of the 4th USSR-Japan Symposium, held at Tbilisi, USSR, August 23–29, 1982, Lectures Notes in Math., vol. 1021, Springer-Verlag (to appear).

DEPARTMENT OF MATHEMATICS, TOYO UNIVERSITY, SAITAMA, JAPAN