AN INFINITE-DIMENSIONAL PRE-HILBERT SPACE NOT HOMEOMORPHIC TO ITS OWN SQUARE

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ABSTRACT. Given an arbitrary infinite-dimensional separable complete linear metric space X , there exists a direct sum decomposition $X = V_0 \oplus V_1$ such that each summand V_i intersects every linearly independent Cantor set in X (this decomposition can be considered as a linear analogue to the classical Bernstein's decomposition into totally imperfect sets).

THEOREM. *Each summand V of such a decomposition is not homeomorphic to its own square, and if $T: V \rightarrow V$ is a linear bounded operator, then either the kernel or the range of T is finite-dimensional.*

In the case of $X = l_2$ this provides an example of a space V with the properties stated in the title, which answers a well-known question, cf. Arhangel'skiĭ [A, Problem 21] and Geoghegan [G, Problem (LS 12)].

1. Bernstein-like direct sum decompositions of infinite-dimensional complete linear separable metric spaces. Let X be an infinite-dimensional separable complete linear metric space (cf. [B-P] for the terminology). In this note we shall consider direct sum decompositions

$$(1) \quad X = V_0 \oplus V_1$$

such that

$$(2) \quad \text{if } C \text{ is a linearly independent Cantor set in } X, \text{ then } V_0 \cap C \neq \emptyset \neq V_1 \cap C;$$

here "Cantor set" means "topologically the Cantor set" and we shall call "Bernstein-like direct sum decomposition of X " any decomposition of X satisfying (1) and (2).

To obtain a Bernstein-like direct sum decomposition of X one can follow a classical Bernstein's reasoning [K₁, I, §36, proof of Theorem 1]:

Let $\{C_\alpha: \alpha < 2^\omega\}$ be all linearly independent Cantor sets in X ,² and let us choose inductively points $x_\alpha, y_\alpha \in C_\alpha$ such that at each stage β of the construction the vectors $\{x_\alpha: \alpha < \beta\} \cup \{y_\alpha: \alpha < \beta\}$ are linearly independent; then let us add to the collection $\{x_\alpha: \alpha < 2^\omega\} \cup \{y_\alpha: \alpha < 2^\omega\}$ a set A to obtain a linear basis in X ; and finally, let V_0 be the linear span of the set $\{x_\alpha: \alpha < 2^\omega\} \cup A$ and let V_1 be the linear span of the set $\{y_\alpha: \alpha < 2^\omega\}$.

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²If there were no such sets in X , (2) would turn simply into an empty condition; in fact, a theorem of Mycielski, see §2, shows that almost every Cantor set in X is linearly independent.

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2. The results.

2.1. THEOREM. *Let X be an infinite-dimensional separable complete linear metric space and let $X = V_0 \oplus V_1$ be a Bernstein-like direct sum decomposition of X (see §1). Then each summand V of the decomposition has the following properties:*

- (i) V is of second category in X ;
- (ii) V is not homeomorphic to its own square $V \times V$, and moreover,
- (iii) if V is homeomorphic to the product $E \times F$ of arbitrary spaces then either E or F contains a nonempty open finite-dimensional subspace;
- (iv) if $T: V \rightarrow V$ is a linear bounded operator then either the kernel or the range of T is finite-dimensional.

2.2. REMARK. If we take $X = l_2$, then the space V described in Theorem 2.1 is a pre-Hilbert space not homeomorphic to its own square, which answers a well-known question, cf. Arhangel'skii [A, Problem 21] and Geoghegan [G, Problem (LS 12)]. Property (iv) shows also that in the pre-Hilbert space V the only complementable linear subspaces are that of finite dimension or codimension.

3. A theorem of Mycielski on independent Cantor sets. The proof of Theorem 2.1 will be based on the following fact, being a fairly simple consequence of a general theorem on independent sets due to Mycielski [M₁], cf., also [M₂] and [K₂].

3.1. LEMMA. *Let A be an analytic set in a separable complete linear metric space X . If A contains an uncountable linearly independent set L , then A contains a linearly independent Cantor set.*

PROOF. Let $f: P \rightarrow A$ be a continuous map of the irrationals P onto A , let M be a subset of P which is mapped by f in a one-to-one way onto L and let Z be the set of all condensation points of M in P [K₁, §18]. Let $R_n = \{(x_1, \dots, x_n) \in Z^n: \text{the vectors } f(x_1), \dots, f(x_n) \text{ are linearly dependent}\}$. Then Z is a completely metrizable space and each R_n is closed in Z^n . Moreover, $M \cap Z$ is dense in Z and if $x_1, \dots, x_n$ are distinct points from M , then $(x_1, \dots, x_n) \notin R_n$, so R_n is nowhere dense in Z^n . Thus a theorem of Mycielski [M₁, Theorem 1] and [M₂, Theorem 1] guarantees an existence of a Cantor set $C \subset Z$ such that if $x_1, \dots, x_n$ are distinct points from C then $(x_1, \dots, x_n) \notin R_n$, i.e. the vectors $f(x_1), \dots, f(x_n)$ are linearly independent. It follows that f maps C homeomorphically onto a linearly independent Cantor set $f(C) \subset A$.

4. An auxiliary proposition. The following, rather special proposition, states a property of the space V which is most essential in the proof of Theorem 2.1(iii) in §5.2 (cf. also Remark 7).

4.1. PROPOSITION. *Let $X = V_0 \oplus V_1$ be a Bernstein-like direct sum decomposition as in Theorem 2.1 and let V be any summand of this decomposition. Assume that G is a G_δ -set in X containing V and that $f: G \rightarrow Y$ is a continuous map onto a metrizable space Y . Then, either the image $f(V)$ is open modulo a first category set in Y , or there exists a point $y \in f(V)$ such that the fiber $f^{-1}(y)$ contains a relatively open nonempty finite-dimensional subset.*

PROOF. At first, in (I) and (II), we shall recall some facts about the Effros Borel structure (the reader can consult Dellacherie [D, §2] or Christensen [Ch] for this topic); the essential part of the proof is the third one.

(I) Let $\mathbf{F}(G)$ be the set of all closed subsets of G and let $\mathbf{B}(G)$ be the Effros Borel structure in $\mathbf{F}(G)$, i.e. the σ -algebra generated by the sets

$$(3) \quad \mathbf{U} = \{F \in \mathbf{F}(G) : F \cap U \neq \emptyset\}, \quad U \text{ being open in } G.$$

Let Z be a compact metrizable extension of X . Then the Effros Borel structure $\mathbf{B}(Z)$ in the hyperspace $\mathbf{F}(Z)$ of Z coincides with the σ -algebra of Borel sets of the topological space $\mathbf{F}(Z)$ endowed with the exponential topology and the map $F \rightarrow \bar{F}$ from $\mathbf{F}(G)$ to $\mathbf{F}(Z)$ (the closure being taken in Z) is $(\mathbf{B}(G), \mathbf{B}(Z))$ -measurable. Since the set of all finite-dimensional compact subsets of Z is in $\mathbf{B}(Z)$ [K₁, Volume 2, Theorem 4, p. 108], and since this is also true for the set of all compact subsets of Z contained in X , the measurability of the map $F \rightarrow \bar{F}$ guarantees that

(4)

$$\mathbf{E} = \{F \in \mathbf{F}(G) : \text{the closure in } X \text{ of } F \text{ is compact and finite-dimensional}\} \in \mathbf{B}(G).$$

(II) Given an open set W in G , let us define a map $\Phi : Y \rightarrow \mathbf{F}(G)$ by the formula

$$\Phi(y) = \overline{f^{-1}(y) \cap W}.$$

Let $\mathbf{A}(Y)$ be the σ -algebra in Y generated by the family of analytic sets; recall, that by a classical theorem of Szpilrajn-Marczewski [K₁, II, §35, Corollary 1], each set in $\mathbf{A}(Y)$ is open modulo a first category set in Y . Let us verify that

$$(5) \quad \Phi \text{ is } (\mathbf{A}(Y), \mathbf{B}(G))\text{-measurable.}$$

Indeed, if $\mathbf{U}$ is a generator of the σ -algebra $\mathbf{B}(G)$ described in (3), then $\Phi^{-1}(\mathbf{U}) = \{y \in Y : \overline{f^{-1}(y) \cap W} \cap U \neq \emptyset\} = f(U \cap W)$, which is an analytic set.

(III) Let $W_1, W_2, \dots$ be a countable base in G and let, for each i ,

$$(6) \quad \Phi_i(y) = \overline{f^{-1}(y) \cap W_i}.$$

It follows from (4) and (5) that

$$(7) \quad E_i = \Phi_i^{-1}(\mathbf{E} \setminus \{\emptyset\}) \in \mathbf{A}(Y),$$

so in particular, each E_i is open modulo a first category set in Y . Let us check that

$$(8) \quad Y \setminus f(V) \subset \bigcup_i E_i.$$

If $y \in Y \setminus f(V)$, then $f^{-1}(y)$ is a G_δ -set in X disjoint from V and hence, by Lemma 3.1 and property (2) of V , the linear space L spanned by the set $f^{-1}(y)$ has the countable linear dimension. Thus L is a union of countably many finite-dimensional linear subspaces, and therefore $f^{-1}(y)$ can be covered by countably many compact finite-dimensional sets $L_1, L_2, \dots$. By the Baire category theorem, one of the sets $L_k \cap f^{-1}(y)$ has the nonempty interior in the space $f^{-1}(y)$, and therefore there exists an element W_i in our base such that $\emptyset \neq f^{-1}(y) \cap W_i \subset L_k$. This means that (see (6) and (4)) $\Phi_i(y) \in \mathbf{E} \setminus \{\emptyset\}$, i.e. $y \in E_i$ (see (7)).

To end the proof it remains now to interpret properly the inclusion (8): either $f(V) = Y \setminus \bigcup_i E_i$ and then the set $f(V)$ is open modulo a first category set in Y , or

else, there exists a point $y \in f(V) \cap E_i$, for some i , and then (see (4), (6) and (7)) $f^{-1}(y) \cap W_i$ is a nonempty open finite-dimensional set in the fiber $f^{-1}(y)$.

5. Proof of Theorem 2.1. Let $X = V_0 \oplus V_1$ be a Bernstein-like direct sum decomposition as in Theorem 2.1 and let V be one of the summands of this decomposition.

5.1. PROOF OF (i). Assume, toward a contradiction, that V is of first category in X . Then there exists a dense G_δ -set in X disjoint from V and, by virtue of Lemma 3.1 and property (2) of V , the linear space spanned by this G_δ -set has the countable linear dimension. But then, the Baire category theorem easily yields an existence of a nonempty open finite-dimensional subset of X —a contradiction with infinite-dimensionality of X .

5.2. PROOF OF (iii). Assume that there exists a homeomorphism of the space V onto the Cartesian product $E \times F$ of arbitrary spaces E, F ; let $\tilde{E}$ and $\tilde{F}$ be completions of E and F , respectively; and let us extend this homeomorphism, using the Lavrentieff theorem [K₁, §31, II], to a homeomorphism $h: G \rightarrow H$ which maps a G_δ -set G in X containing V onto a G_δ -set H in the product $\tilde{E} \times \tilde{F}$ containing $E \times F$. Notice that $h(V) = E \times F$.

Let us assume that the space F does not contain any nonempty open finite-dimensional sets. Let $p: \tilde{E} \times \tilde{F} \rightarrow \tilde{E}$ be the projection, let $Y = p(H)$ and let $f = p \circ h: G \rightarrow Y$. Let us apply Proposition 4.1 to the map f .

The second possibility in the alternative asserted in Proposition 4.1 cannot occur, as for each $y \in f(V) = E$ we have $f^{-1}(y) = h^{-1}(\{y\} \times \tilde{F} \cap H)$, thus F embeds topologically as a dense subspace in $f^{-1}(y)$ and hence there is no nonempty open finite-dimensional sets in the space $f^{-1}(y)$.

Therefore, the first part of this alternative must be true, i.e. E is open modulo a first category set in Y . Now, Y being an analytic set, Y is open modulo a first category set in $\tilde{E}$ (the Szpilrajn-Marczewski theorem [K₁, II, §35]) and hence E is open modulo a set of first category in $\tilde{E}$. Moreover, E is a space of second category, being a factor of the second category space V (see §5.1), and so E contains a G_δ -subset A of $\tilde{E}$ of second category in $\tilde{E}$. Fix an $a \in F$ and put $A' = h^{-1}(A \times \{a\})$, $E' = h^{-1}(E \times \{a\})$. The set A' is a G_δ -set in X contained in V , hence by Lemma 3.1 and property (2) of V , the linear span of A' has the countable linear dimension, so A' is covered by countably many closed in X finite-dimensional sets L_i . The restriction $f|_{E'}$ being a homeomorphism onto E , it follows that the closed finite-dimensional subsets $f(L_i \cap E')$ of E cover the set A . Finally, the Baire category theorem applied to the completely metrizable space A of second category in E yields easily an existence of a nonempty open finite-dimensional subspace of E .

5.3. PROOF OF (iv). (We use here some improvements of the original reasoning, due to T. Dobrowolski.) Let $T: V \rightarrow V$ be a linear bounded operator and let f be the extension of T to a linear operator mapping the space X into itself.

If there exists an $y \in f(X) \setminus V$, then the space $f^{-1}(y) = x + \text{Ker } f$ (x being any vector with $f(x) = y$) is disjoint from V and hence $\text{Ker } f$ has the countable linear dimension, by Lemma 3.1 and property (2) of V , i.e. $\text{Ker } f$ is finite-dimensional, being a complete space. Thus in this case $\text{Ker } T \subset \text{Ker } f$ is finite-dimensional.

The other alternative is that $f(X) \subset V$. Then $f(X)$ is an analytic linear subspace of V , and again by Lemma 3.1 and (2), $f(X)$ has the countable linear dimension. But

the Baire category theorem applied to X yields then easily that $f(X)$ is, in fact, finite-dimensional and so is the range of the operator T .

6. Remark. If one is only interested in a construction of a space V with the properties stated in the title, then one can replace the main part of the proof given in §5.2 by the following simpler argument.

6.1. LEMMA. *Let Z be a compact extension of the space X and let H be a G_δ -set in $X \times Z$ containing the square $V \times V$ of the space V . Then the set H^* of all $x \in X$ such that the vertical section $H(x)$ of H at x contains a topological copy of the Hilbert cube I^ω is an analytic set.*

PROOF. Let $(X \times Z) \setminus H = F_1 \cup F_2 \cup \dots$, F_i being a closed set in $X \times Z$, let $\mathbf{E}$ be the space of topological embeddings of I^ω into Z endowed with the compact-open topology, and let $\mathbf{U}_n = \{(x, h) \in X \times \mathbf{E} : h(I^\omega) \cap F_n(x) = \emptyset\}$, $F_n(x)$ being the vertical section of F_n at x . Then the sets $\mathbf{U}_n$ are open and H^* is the projection of the G_δ -set $\bigcap_n \mathbf{U}_n$ in $X \times \mathbf{E}$ onto the X -axis.

Now, let us adopt the notation of §5.2 with $E = F = V$ and $\tilde{E} = \tilde{F} = X$, where we assume also that X is a Banach space.

If $x \notin V$, then $h^{-1}(\{x\} \times X \cap H)$ is a G_δ -set in X disjoint from V and hence it does not contain a topological copy of I^ω , by Lemma 3.1 and (2), and so the Hilbert cube does not embed into the vertical section $H(x)$. On the other hand, if $x \in V$, then $H(x)$ is a completely metrizable space containing an infinite-dimensional normed space V , and thus the Hilbert cube embeds into $H(x)$, cf. [B-P, Chapter VIII, Theorem 3.1].

It follows that V coincides with the analytic set H^* defined in Lemma 7.1. This however, contradicts Lemma 3.1 and property (2) of the space V .

6.2. REMARK. One can consider the above reasoning as a reminiscence of an old idea in the theory of analytic sets going back to Mazurkiewicz and Sierpiński, see Kuratowski [K₁, VII, §43, Proof of Corollary 3].

ADDED IN PROOF. W. Marciszewski recently constructed in a paper *A pre-Hilbert space without any continuous map onto its own square*, Bull. Polon. Acad. Sci. (to appear) a space with the properties indicated in the title.

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