GROUP ACTIONS ON ASPHERICAL $A_k(N)$ -MANIFOLDS WITH NONZERO EULER CHARACTERISTICS

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ABSTRACT. By an aspherical $A_k(N)$ -manifold, we mean a compact manifold M together with a map f from M into an aspherical complex N such that $f^*: H^k(N; Q) \rightarrow H^k(M; Q)$ is nontrivial. In this paper we study the fixed point set, degree of symmetry, semisimple degree of symmetry and torus degree of symmetry of M with the Euler characteristic of M nonzero.

Suppose M^m is a compact connected m -manifold. The degree of symmetry $N_T(M)$ (resp. *semisimple degree of symmetry* $N_T^s(M)$, and *torus degree of symmetry* $T_t(M)$) is defined as the supremum of the dimensions of all compact Lie (resp. compact semisimple Lie, and torus) groups which can act effectively on M . Following [10], we shall say that M is an *aspherical $A_k(N)$ -manifold*, k a nonnegative integer, if N is an aspherical complex and there exists a continuous map $f: M \rightarrow N$ such that $f^*: H^k(N; Q) \rightarrow H^k(M; Q)$ is nontrivial. We shall always assume that $\pi_1(N)$ is torsion-free. For an aspherical $A_k(N)$ -manifold M , it is known that (cf. [6, 10, 14])

$$(1) \quad N_T^s(M) \leq \langle m - k + 1 \rangle,$$

and

$$(2) \quad N_T(M) \leq k + \langle m - k + 1 \rangle,$$

where $\langle s \rangle$ denotes $\dim SO(s)$.

Throughout this paper, the Alexander-Spanier cohomology with compact supports will be used.

In this paper we shall study the aspherical $A_k(N)$ -manifold M with nonzero Euler characteristic $\chi(X)$. For instance, under this hypothesis, (2) can be improved as

$$(3) \quad N_T(M) \leq \langle m - k + 1 \rangle.$$

The estimate (3) also holds in case N is a Riemannian manifold of negative curvature (see [6, 10, 14]) if $k > 1$.

THEOREM 1. *Let $f: M \rightarrow N$ be an aspherical $A_k(N)$ -manifold.*

(a) *If a torus group G acts on M with fixed point set F nonempty, then at least one component of F is an aspherical cohomology $A_k(N)$ -manifold.*

(b) *If $\chi(X) \neq 0$, then*

$$(4) \quad T_t(M) \leq [(m - k)/2],$$

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and

$$(5) \quad N_T(M) \leq \langle m - k + 1 \rangle,$$

where $[s]$ denotes the largest integer less than or equal to s .

PROOF. (a) Let $F = \bigcup_{j=1}^s F_j$ and $f_j = f|_{F_j}: F_j \rightarrow N$. By the Smith theory [1], each F_j is a cohomology manifold. Suppose that

$$f_j^*: H^k(N; Q) \rightarrow H^k(F_j; Q)$$

is trivial for all $j = 1, \dots, s$. We shall get a contradiction. Let $x \in F$ and $i: G \rightarrow M$ be the orbit map defined by $i(g) = gx$. As $i_*\pi_1(G) = 0$, it follows from [6] that $\pi_1(M/G) \cong \pi_1(M)/H$, where H is a normal subgroup of $\pi_1(M)$ generated by elements of finite order. Hence if $\pi: M \rightarrow M/G$ is the natural projection we have

$$\text{Ker}\{\pi_*: \pi_1(M) \rightarrow \pi_1(M/G)\} \subset \text{Ker}\{f_*: \pi_1(M) \rightarrow \pi_1(N)\}.$$

Thus, there exists a map $h: M/G \rightarrow N$ such that $h\pi$ is homotopic to f by [6]. Let $q: F \rightarrow M/G$ and $p: F \rightarrow M$ be inclusions. It is easy to see that $q^*h^* = p^*\pi^*h^* = p^*f^* = \sum_{j=1}^s f_j^* = 0$ in degree k . Hence if $\bar{z} \in H^k(N; Q)$ be such that $f^*(\bar{z}) = z \neq 0$, then $q^*h^*(\bar{z}) = 0$. From the exact cohomology sequence of the pair $(M/G, F)$, there exists $z' \in H^k((M - F)/G; Q)$ such that $j'^*(z') = h^*(\bar{z})$, where j' is an inclusion. Consider the following commutative diagram

$$\begin{array}{ccccc} H_G^k(F; Q) & \xleftarrow{\sum i_j^*} & H_G^k(M; Q) & \xleftarrow{j^*} & H_G^k(M - F; Q) \\ & & \downarrow i^* & \nearrow j'^* & \downarrow \pi_1^* \\ & & H^k(M/G; Q) & \xleftarrow{j'^*} & H^k\left(\frac{M - F}{G}; Q\right) \\ & & \downarrow \pi^* & & \downarrow \\ & & H^k(M; Q) & \xleftarrow{\quad} & H^k(M - F; Q) \end{array}$$

where π_1 is induced by natural projection and i inclusion. It follows that $\sum_j i_j^*(j^*\pi_1^*z') = 0$ and $i^*(j^*\pi_1^*z') = z$. Hence $j^*\pi_1^*z'$ is $H^*(B_G; Q)$ -free. This contradicts to the following Borel Localization Theorem [7]:

$$\sum_{j=1}^s S^{-1}i_j^*: S^{-1}H_G^*(M; Q) \cong S^{-1}H_G^*(F; Q),$$

where $S = H^*(B_G; Q) - \{0\}$.

(b) Let G be a torus group acting effectively on M . Since $\chi(F) = \chi(M) \neq 0$, the fixed point set F is not empty. Hence by [8], $\dim G \leq [(m - \dim F)/2]$ but $\dim F \geq k$ by (a). The result (4) follows.

To prove (5), let G be a compact connected Lie group acting effectively on M and $\dim G = N_T(M)$. Expressed G as $G = (T^s \oplus S)/N$, where T^s is a torus group of rank s , S semisimple and N a finite normal subgroup. According to (1) we have $\dim S \leq \langle m - k + 1 \rangle$. Let $r = \text{rank } S$. By (4), we get $s + r \leq [(m - k)/2]$. Hence

$$(6) \quad \dim G \leq [(m - k)/2] - r + \dim S.$$

We shall divide the proof into three cases.

Case 1. $r = 0$. Then $\dim G = s \leq [(m - k)/2] \leq \langle m - k + 1 \rangle$.

Case 2. $r > 0$, and $\dim S = \langle m - k + 1 \rangle$. In this case it can be shown (cf. [9]) that S is locally isomorphic to $SO(m - k + 1)$, hence

$$s + r = s + [(m - k + 1)/2] \leq [(m - k)/2].$$

It follows that $s = 0$.

Case 3. $r > 0$, and $\dim S < \langle m - k + 1 \rangle$.

Let $S(x)$ be a principal orbit. We know that $\dim S(x) \leq m - k$ (cf. [6, 10]). Let

$$W = \begin{cases} S(x) & \text{if } \dim S(x) = m - k, \\ S(x) \times S^{m-k-\dim S(x)} & \text{with } S \text{ acting trivially on} \\ & S^{m-k-\dim S(x)} \text{ if } \dim S(x) < m - k. \end{cases}$$

Consider the action of S on W . By gap theorem [12], we have

$$(7) \quad \dim S \leq \langle m - k \rangle + 1$$

with the following three exceptions:

- (i) $m - k = 4$, $W \approx CP^2$, and $S \approx SU(3)$;
- (ii) $m - k = 6$, $W \approx S^6$ or RP^6 , and $S \approx G_2$;
- (iii) $m - k = 10$, $W \approx CP^5$, and $S \approx SU(6)$. It follows from (6) and (7) that

$$\begin{aligned} \dim G &\leq (m - k)/2 - r + \langle m - k \rangle + 1 \\ &\leq (m - k)^2/2 < \langle m - k + 1 \rangle. \end{aligned}$$

This also holds for the three exceptional cases. For instance, (ii) will imply that

$$\dim G \leq 3 - 2 + 14 < \langle 7 \rangle = 21.$$

This completes the proof of the theorem.

COROLLARY 1. Let M be an aspherical $A_k(N)$ -manifold with $\chi(M) \neq 0$.

- (a) If $k = m$, or $m - 1$, then $N_T(M) = 0$.
- (b) If $k = m - 2$ and $\chi(M)$ odd, then $N_T(M) \leq 1$.
- (c) If $k = m - 3$, then $N_T(M) \leq 3$.

PROOF. It suffices to show (b). As in [9], if $N_T(M) = 3$, we can show that M is homeomorphic to $S^2 \times_{Z_2} P$, P an aspherical $A_{m-2}(N)$ -manifold. It follows that $\chi(M) = 2\chi(P)$.

As an immediate corollary to Corollary 1 we have

COROLLARY 2. Let M be a closed connected oriented manifold and M_1 a closed aspherical $A_k(N)$ -manifold and $\chi(M) + \chi(M_1) \neq 2$.

- (a) If $k = m$, or $m - 1$, then $N_T(M \# M_1) = 0$.
- (b) If $k = m - 2$, and $\chi(M) + \chi(M_1)$ is odd, then $N_T(M \# M_1) \leq 1$.
- (c) If $k = m - 3$, $N_T(M \# M_1) \leq 3$.

Theorem 1(a) was proved in [10] for the case of smooth action and Pontrjagin number of f nonzero. Corollary 1(a) for $M = N$ and f the identity was proved by Conner and Montgomery in [4] (cf. [5]). The special case of Corollary 2(a) for $M_1 = T^m$ was verified in [3, 9, 13]. Corollary 1(c) is a generalization of [9, Corollary 3.11].

From the proof of Theorem 1(a), we also obtain the following

COROLLARY 3. *Let L be a torsion-free group. Let G be a torus group acting on a compact connected manifold M with nonempty fixed point set. Then the projection $\pi: M \rightarrow M/G$ induces surjection*

$$\pi^*: H^1(M/G; L) \rightarrow H^1(M; L).$$

For $L = Q$, this corollary was proved in [2].

For a compact connected manifold M with $\pi_1(M)$ torsion-free, it has a natural $A_k(M)$ structure, where k is the aspherical index $a(M)$ of M which is defined as follows. Let $f: M \rightarrow K(\pi_1(M), 1)$ be a classifying map and $a(M)$ be the largest integer k such that $f^*: H^k(K(\pi_1(M), 1); Q) \rightarrow H^k(M; Q)$ is nontrivial. For examples, $a(S^{m-k} \times T^k) = k$ and $a(M) = m$ if M is a closed oriented aspherical m -manifold.

From (1), (2) and Theorem 1 we have

THEOREM 2. *Let M be a compact connected m -manifold. Then*

- (a) $N_T^s(M) \leq \langle m - a(M) + 1 \rangle$.
- (b) $N_T(M) \leq a(M) + \langle m - a(M) + 1 \rangle$.
- (c) *If a torus G acts on M with nonempty fixed point set F , then at least one component of F , say F_0 , satisfies $H^{a(M)}(F_0; Q) \neq 0$.*
- (d) *If $\chi(M; Q) \neq 0$, then*

$$T_i(M) \leq [(m - a(M))/2] \quad \text{and} \quad N_T(M) \leq \langle m - a(M) + 1 \rangle.$$

Similarly, we also have results corresponding to both Corollary 1 and Corollary 2.

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