BANACH SPACES THAT HAVE NORMAL STRUCTURE AND ARE ISOMORPHIC TO A HILBERT SPACE

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ABSTRACT. We prove that given a Hilbert space $(E, \|\cdot\|)$, and $|\cdot|$ a norm on E such that for all $x \in E$, $1/\beta|x| \leq \|x\| \leq |x|$ for some β , if $1 \leq \beta < \sqrt{2}$, then $(E, |\cdot|)$ satisfies a convexity property from which normal structure follows.

1. Introduction. A Banach space E is said to have normal structure if for each bounded, closed and convex subset C of E , consisting of more than one point, there is an $x \in C$ such that

$$\sup\{\|x - y\| : y \in C\} < \text{diam}(C) \equiv \sup\{\|y_1 - y_2\| : y_1, y_2 \in C\}.$$

In [4] it was proved that if E has normal structure, $C \subseteq E$ is a nonempty weakly compact convex set, and $T: C \rightarrow C$ is a mapping such that for all $x, y \in C$, $\|Tx - Ty\| \leq \|x - y\|$, then T has a fixed point in C .

For $r \geq 1$ let E_r be the space l_2 renormed by

$$|x|_r := \max\{\|x\|_2, r\|x\|_\infty\},$$

where $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the l_2 and l_∞ norms, respectively. It is known from [1] that E_r has normal structure when $r < \sqrt{2}$.

We use the idea of multidimensional volumes and their related convexity moduli to prove E_r satisfies a convexity property that implies this result. The notion of volumes in Banach spaces was introduced by Silverman and its use in defining moduli of convexity was introduced in [5]. Roughly speaking, the modulus of k -rotundity, $\delta_k(\epsilon)$, measures the depth below the surface of the unit sphere of the centroid of a simplex of $k + 1$ norm-1 vectors enclosing a k -dimensional volume larger than ϵ . In symbols,

$$A(x_1, \dots, x_{k+1}) \geq \epsilon$$

implies that

$$\|(x_1 + \dots + x_{k+1})/(k+1)\| \leq 1 - \delta_k(\epsilon).$$

Here $A(x_1, \dots, x_{k+1})$ denotes the enclosed volume. In case $k = 1$, $A(x_1, x_2) = \|x_1 - x_2\|$ and $\delta_1(\epsilon)$ is the usual modulus of convexity. In all cases

$$D(\|\cdot\|, x_1, \dots, x_{k+1}) \leq A(x_1, \dots, x_{k+1}).$$

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Here

$$D(\|\cdot\|, x_1, \dots, x_{k+1}) \equiv \|x_k - x_{k+1}\| \cdot \text{dist}(x_{k-1}, [x_k, x_{k+1}]) \\ \cdot \dots \cdot \text{dist}(x_1, [x_2, \dots, x_{k+1}])$$

where $[x_{i+1}, \dots, x_{k+1}]$ is the affine span of the vectors $x_{i+1}, \dots, x_{k+1}$ and

$$\text{dist}(x_i, [x_{i+1}, \dots, x_{k+1}]) = \inf\{\|x_i - x\| : x \in [x_{i+1}, \dots, x_{k+1}]\}.$$

For a Hilbert space the inequality is always equality.

A connection between these moduli and normal structure of a Banach space E was given in [3], namely

LEMMA. *Suppose that for some $\delta > 0$ and some $0 < \varepsilon < 1$ there is an integer m such that for all norm-1 $x_1, \dots, x_m \in E$, if $\|(x_1 + x_2 + \dots + x_m)/m\| > 1 - \delta$ then $D(\|\cdot\|, x_1, \dots, x_m) < \varepsilon$.*

Then E is super-reflexive and has normal structure.

2. The result.

THEOREM. *Let $(E, \|\cdot\|)$ be a Hilbert space, and let $|\cdot|$ be a norm on E such that for all $x \in E$, $1/\beta|x| \leq \|x\| \leq |x|$ for some β , $1 \leq \beta < \sqrt{2}$. Given $\varepsilon > 0$, there exists $\delta > 0$ and M , a positive integer, such that for $m \geq M$, if $x_1, \dots, x_m \in E$, $|x_1|, \dots, |x_m| \leq 1$, and $|(x_1 + \dots + x_m)/m|^2 > 1 - \delta$, then $D(|\cdot|, x_1, \dots, x_m) < \varepsilon$.*

The proof requires some preliminary results.

LEMMA 1. *Given k , a positive integer and $\beta > 0$, let f, g be the functions from R^k into R defined by*

$$f(x_1, \dots, x_k) = \frac{1}{k+1} \sum_{i=1}^k \frac{i}{i+1} (x_{k+1-i})^2, \\ g(x_1, \dots, x_k) = \beta - \prod_{i=1}^k x_i, \quad (x_1, \dots, x_k) \in R^k.$$

Then $f(x) \geq (k/(k+1))(\beta^{2/k}/(k+1)^{1/k})$ whenever $x \in R^k$ and $g(x) = 0$.

PROOF. Let $w \in \Omega \equiv \{x \in R^k : g(x) = 0\}$ and $y = f(w) > 0$. Then $\hat{\Omega} \equiv f^{-1}([0, y]) \cap \Omega$ is nonempty and compact and, thus, there exists $x^* \in \hat{\Omega}$, a global minimum point of f over $\hat{\Omega}$. Also, if $z \in \Omega \setminus f^{-1}([0, y])$ then $f(z) > y$ so that x^* is a global minimum point of f over Ω .

With $x^* = (b_1, \dots, b_k)$ then $\prod_{i=1}^k b_i = \beta > 0$. Thus $\nabla g(x^*) \neq 0$, where ∇g is the gradient of g . It now follows, by Lagrange's theorem, that for some $\lambda \in R$,

$$\nabla f(x^*) = \lambda \nabla g(x^*).$$

So, for each i , $1 \leq i \leq k$,

$$\frac{2}{k+1} \frac{k+1-i}{k+2-i} b_i = \lambda \prod_{\substack{j=1 \\ j \neq i}}^k b_j$$

or

$$\frac{2}{k+1} \frac{k+1-i}{k+2-i} b_i^2 - \lambda \beta = 0.$$

Thus, for each i , $1 \leq i \leq k$, $((k+1)/2)\lambda\beta = ((k+1-i)/(k+2-i))b_i^2$, and

$$\left[\frac{k+1-i}{k+2-i} b_i^2 \right]^k = \prod_{j=1}^k \frac{k+1-j}{k+2-j} b_j^2 = \frac{1}{k+1} \beta^2.$$

Therefore, $((k+1-i)/(k+2-i))b_i^2 = \beta^{2/k}/(k+1)^{1/k}$ for each i , $1 \leq i \leq k$.

So, given $x \in \Omega$,

$$\begin{aligned} f(x) &\geq f(x^*) = \frac{1}{k+1} \sum_{i=1}^k \frac{i}{i+1} b_{k+1-i}^2 \\ &= \frac{1}{k+1} \sum_{i=1}^k \frac{k+1-i}{k+2-i} b_i^2 = \frac{k}{k+1} \frac{\beta^{2/k}}{(k+1)^{1/k}}. \end{aligned}$$

LEMMA 2. *Let $(E, \|\cdot\|)$ be a Hilbert space and k any positive integer. Given $x_1, \dots, x_{k+1} \in E$, $\|x_1\|, \dots, \|x_{k+1}\| \leq 1$, then*

$$\left\| \frac{x_1 + \dots + x_{k+1}}{k+1} \right\|^2 \leq 1 - \frac{1}{k+1} \sum_{i=1}^k \frac{i}{i+1} \left\| x_{k+1-i} - \frac{\sum_{j=k+2-i}^{k+1} x_j}{i} \right\|^2.$$

PROOF. Since $(E, \|\cdot\|)$ is a Hilbert space, given $x, y \in E$, then

$$\left\| \frac{1}{k+1} x + \frac{k}{k+1} y \right\|^2 = \frac{1}{k+1} \|x\|^2 + \frac{k}{k+1} \|y\|^2 - \frac{k}{(k+1)^2} \|x - y\|^2.$$

In particular, given $x_1, \dots, x_{k+1} \in E$,

$$\begin{aligned} \left\| \frac{x_1 + \dots + x_{k+1}}{k+1} \right\|^2 &= \frac{1}{k+1} \|x_1\|^2 + \frac{k}{k+1} \left\| \frac{x_2 + \dots + x_{k+1}}{k} \right\|^2 \\ &\quad - \frac{k}{(k+1)^2} \left\| x_1 - \frac{x_2 + \dots + x_{k+1}}{k} \right\|^2. \end{aligned}$$

The proof of the lemma now follows by induction on k .

LEMMA 3. *Let $(E, \|\cdot\|)$ be a Hilbert space and k a positive integer. If $x_1, \dots, x_{k+1} \in E$, $\|x_1\|, \dots, \|x_{k+1}\| \leq 1$, $D(\|\cdot\|, x_1, \dots, x_{k+1}) \geq \varepsilon > 0$, then*

$$\|(x_1 + \dots + x_{k+1})/(k+1)\|^2 \leq 1 - (k/(k+1))(\varepsilon^{2/k}/(k+1)^{1/k}).$$

PROOF. Since $D(\|\cdot\|, x_1, \dots, x_{k+1}) \geq \varepsilon$ then $D(\|\cdot\|, x_1, \dots, x_{k+1}) = \beta$, where $\beta \geq \varepsilon$. Let $d_i = \text{dist}(x_i, [x_{i+1}, \dots, x_{k+1}])$, for each i , $1 \leq i \leq k$. By Lemma 2, with f as defined in Lemma 1, it follows that

$$\begin{aligned} \left\| \frac{x_1 + \dots + x_{k+1}}{k+1} \right\|^2 &\leq 1 - \frac{1}{k+1} \sum_{i=1}^k \frac{i}{i+1} \left\| x_{k+1-i} - \frac{\sum_{j=k+2-i}^{k+1} x_j}{i} \right\|^2 \\ &\leq 1 - \frac{1}{k+1} \sum_{i=1}^k \frac{i}{i+1} d_{k+1-i}^2 = 1 - f(d_1, \dots, d_k). \end{aligned}$$

However, $\prod_{i=1}^k d_i = D(\|\cdot\|, x_1, \dots, x_{k+1}) = \beta$. So, by Lemma 1,

$$f(d_1, \dots, d_k) \geq (k/(k+1))(\beta^{2/k}/(k+1)^{1/k}),$$

and, therefore,

$$\|(x_1 + \dots + x_{k+1})/(k+1)\|^2 \leq 1 - (k/(k+1))(\epsilon^{2/k}/(k+1)^{1/k}).$$

REMARK. Extending these ideas [2] gives the exact value of the modulus of k -rotundity of a Hilbert space, e.g.

$$\delta_k(\epsilon) = 1 - \left[1 - \frac{k}{k+1} \frac{\epsilon^{2/k}}{(k+1)^{1/k}} \right]^{1/2}.$$

PROOF OF THE THEOREM. Choose $\eta > 0$ so that $\beta^2 + \eta < 2$. Given any $\epsilon > 0$, k a positive integer, let $\Delta_k(\epsilon) = (k/(k+1))(\epsilon^{2/k}/(k+1)^{1/k})$. Since $\lim_{k \rightarrow \infty} \Delta_k(\epsilon) = 1$, select $M > 1$ so large that $\Delta_{m-1}(\epsilon) > 1 - \eta$ whenever $m \geq M$.

Now, let $\delta = 2 - \beta^2 - \eta$, and suppose $m \geq M$, $|(x_1 + \dots + x_m)/m|^2 > 1 - \delta$, $|x_1|, \dots, |x_m| \leq 1$, while $D(\|\cdot\|, x_1, \dots, x_m) \geq \epsilon$. Then $\|x_1\|, \dots, \|x_m\| \leq 1$ and

$$D(\|\cdot\|, x_1, \dots, x_m) \geq (1/\beta)^{m-1} \cdot D(\|\cdot\|, x_1, \dots, x_m) \geq (1/\beta)^{m-1} \epsilon.$$

It follows from Lemma 3 that

$$\left\| \frac{x_1 + \dots + x_m}{m} \right\|^2 \leq 1 - \Delta_{m-1} \left(\left(\frac{1}{\beta} \right)^{m-1} \epsilon \right).$$

However,

$$\begin{aligned} 1 - \delta &< \left| \frac{x_1 + \dots + x_m}{m} \right|^2 \leq \beta^2 \left\| \frac{x_1 + \dots + x_m}{m} \right\|^2 \\ &\leq \beta^2 \left(1 - \Delta_{m-1} \left(\left(\frac{1}{\beta} \right)^{m-1} \epsilon \right) \right) = \beta^2 \left(1 - \left(\frac{1}{\beta} \right)^2 \Delta_{m-1}(\epsilon) \right) \\ &= \beta^2 - \Delta_{m-1}(\epsilon) < \beta^2 + \eta - 1. \end{aligned}$$

This contradicts the definition of δ . Therefore, $D(\|\cdot\|, x_1, \dots, x_m) < \epsilon$. Q.E.D.

The result proven in [1] now follows from this theorem and the lemma mentioned in the introduction.

COROLLARY. Let $(E, \|\cdot\|)$ be a Hilbert space and let $|\cdot|$ be a norm on E such that for all $x \in E$, $1/\beta|x| \leq \|x\| \leq |x|$ for some β , $1 \leq \beta < \sqrt{2}$. Then $(E, |\cdot|)$ has normal structure.

Note that the Theorem is sharp because Baillon and Schöneberg proved that E_r fails to have normal structure for $r \geq \sqrt{2}$.

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