A TOPOLOGICAL SPACE WITHOUT A COMPLETE QUASI-UNIFORMITY

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ABSTRACT. We show that an example of Burke and van Douwen has no complete quasi-uniformity. Moreover, we show that it is almost finitely-fully normal but not almost $\aleph_0$ -fully normal.

0. Introduction. Every topological space admits a quasi-uniformity. The problem whether every topological space admits a complete quasi-uniformity is considered in [3, Problem C], where an example is given of a T_1 -space that admits a complete, but no convergence complete, quasi-uniformity. In this note we show that a locally compact separable normal M -space of D. K. Burke and E. K. van Douwen admits no complete quasi-uniformity, thereby answering an old question in the theory of quasi-uniform spaces. Moreover, we show that this space is an almost finitely-fully normal countably paracompact space that is not almost $\aleph_0$ -fully normal. It is interesting to compare these results with the recent results of K. P. Hart [4] that M. E. Rudin's Dowker space is both orthocompact and finitely-fully normal; for it follows readily from Hart's results that, while Rudin's space is not almost $\aleph_0$ -fully normal, it does admit a complete quasi-uniformity.

1. Definitions and a lemma. A *quasi-uniformity* on a set X is a filter $\mathcal{U}$ on $X \times X$ such that (a) each member of $\mathcal{U}$ is a reflexive relation on X , and (b) if $U \in \mathcal{U}$ then $V \circ V \subset U$ for some $V \in \mathcal{U}$. The pair $(X, \mathcal{U})$ is called a *quasi-uniform space*. A filter $\mathcal{F}$ on $(X, \mathcal{U})$ is a *Cauchy filter* provided that for each $U \in \mathcal{U}$ there exists $p \in X$ so that $U(p) \in \mathcal{F}$, and $(X, \mathcal{U})$ is said to be *complete* provided that every Cauchy filter has a cluster point. The topology $\tau(\mathcal{U}) = \{G \subset X: \text{for each } x \in G \text{ there is } U \in \mathcal{U} \text{ with } U(x) \subset G\}$ is called the *topology induced* by $\mathcal{U}$. A topological space (X, τ) *admits* $\mathcal{U}$ provided that τ is the topology induced by $\mathcal{U}$. Let (X, τ) be a topological space and let $\mathcal{B}$ be the collection of reflexive transitive relations V on X for which $V(x) \in \tau$ for all $x \in X$. Then $\mathcal{B}$ is a filterbase for a quasi-uniformity $\mathcal{U}$. Moreover, using the observation of W. J. Pervin [8] that for each open set G , $G \times G \cup (X \setminus G) \times X \in \mathcal{U}$, we see that (X, τ) admits $\mathcal{U}$. It is known that this quasi-uniformity $\mathcal{U}$ is complete if and only if every ultrafilter on X without a cluster point has a closure-preserving subcollection without a cluster point [3, p. 59]. Consequently, a

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regular space that is either almost real-compact or weakly orthocompact admits a complete quasi-uniformity.

Throughout, if $\mathcal{R}$ is a cover of a space X , and A is a subset of some member of $\mathcal{R}$, we say that A is a *refiner* of $\mathcal{R}$. A space X is *almost finitely-fully normal* (*almost $\aleph_0$ -fully normal*) [6] provided that if $\mathcal{C}$ is an open cover of X there is an open refinement $\mathcal{R}$ of $\mathcal{C}$ with the property that if M is a finite (countable) set and M is a refiner of $\{st(x, \mathcal{R}) \mid x \in X\}$, then M is also a refiner of $\mathcal{C}$.

We begin with a slight extension of a result of G. Aquaro [1] and K. Morita [7, Lemma 4.3].

LEMMA 1. *A space X is almost finitely-fully normal (almost $\aleph_0$ -fully normal) if and only if X is normal, and for each open cover $\mathcal{C}$ of X there is a locally finite open cover $\mathcal{R}$ of X so that if M is a refiner of $\mathcal{R}$ and M is a finite (countable) set, then M is a refiner of $\mathcal{C}$.*

2. The example. The example X under consideration is described completely in [2]. For our purposes it is enough to know the following details: The ground set is $\mu \cup (\omega \times \omega)$, μ and ω considered disjoint, where μ is a regular cardinal, and there is a collection $F = \{f_\alpha : \alpha \in \mu\}$ such that

- (0) for each $\alpha \in \mu$, $f_\alpha \in {}^\omega \omega$,
- (1) each f_α is nondecreasing,
- (2) $f_\alpha < {}^* f_\beta$ if $\alpha < \beta$,
- (3) there is no $g \in {}^\omega \omega$ so that $f_\alpha \leqslant {}^* g$ for all $\alpha \in \mu$.

(As usual, we say $f < {}^* g$ provided that for all but finitely many $n \in \omega$, $f(n) < g(n)$.)

The points of $\omega \times \omega$ are isolated, and basic open sets about $\alpha \in \mu$ with $0 \leqslant \beta < \alpha < \mu$ and $m \in \omega$ are of the form

$$U(\alpha, \beta, m) = (\beta, \alpha] \cup \{\langle k, n \rangle : k \geqslant m \text{ and } f_\beta(k) < n \leqslant f_\alpha(k)\}.$$

If $\alpha \in \mu$ and $\alpha = 0$, basic open sets about α are of the form

$$U(0, \beta, m) = \{0\} \cup \{\langle k, n \rangle : k \geqslant m, n \leqslant f_0(k)\} \quad \text{where } m \in \omega.$$

LEMMA 2. *If S is a cofinal subset of μ , then $A = \{k \in \omega : \langle f_s(k) \rangle_{s \in S} \text{ is eventually bounded}\}$ is an initial segment of ω .*

PROOF. Since each f_α is nondecreasing, we note that if $a < b < \omega$ and $b \in A$, then $a \in A$. Assume $A = \omega$. Then for all $n \in \omega$ there are $s_n \in S$ and $k_n \in \omega$ such that if $s \in S$ and $s > s_n$ then $f_s(n) < k_n$. Define $g: \omega \rightarrow \omega$ by $g(n) = k_n$. There is a tail S' of S such that if $s \in S'$ then $f_s(n) < g(n)$ for all $n \in \omega$. Let $f_\alpha \in F$. There is an $s \in S'$ with $\alpha < s$; by (2), $f_\alpha \leqslant {}^* f_s$, and thus $f_\alpha \leqslant g$. We have shown that (3) fails—a contradiction. ■

X admits no complete quasi-uniformity. For each $x \in \mu$ and $m \in \omega$, set $F(x, m) = \{\langle k, n \rangle : k \geqslant m \text{ and } f_x(k) < n\}$ and let $\mathcal{F}$ be the filter for which $\{F(x, m) : x \in \mu \text{ and } m \in \omega\}$ is a filter base. Clearly no point of $\omega \times \omega$ is a cluster point of $\mathcal{F}$. If $\rho \in \mu$ and $m \in \omega$, then $U(\rho, 0, 0) \cap F(\rho, m) = \emptyset$. Therefore $\mathcal{F}$ is a filter without a cluster point. We show that $\mathcal{F}$ is a Cauchy filter with respect to each quasi-uniformity that X admits. Let $\mathcal{V}$ be such a quasi-uniformity, let $V \in \mathcal{V}$ and let $W \in \mathcal{V}$ so $W^2 \subset V$. For

each $x \in \mu$ choose $\beta_x \in \mu$ and $m_x \in \omega$ so that $U(x, \beta_x, m_x) \subset W(x)$. By the Pressing-Down Lemma, there is a cofinal subset S of μ , $\beta \in \mu$ and $j \in \omega$ so that, for all $s \in S$, $U(s, \beta, j) \subset W(s)$. We note that $\beta < s$ for each $s \in S$. Set $A' = \{k \in \omega: \langle f_s(k) \rangle_{s \in S} \text{ is bounded}\}$. By Lemma 2, A' is finite. There is $e \in \omega$ so that if $k \geq e$ then $\{f_s(k): s \in S\}$ is unbounded. Let $r = \max\{e, j\}$. For each $k \geq r$, define a function $a_k: \omega \rightarrow S$ by letting $a_k(n)$ be the least ordinal $\alpha \in S$ so $f_\alpha(k) \geq n$. Let $\gamma = \sup\{a_k(n): k \geq r, n \in \omega\}$; there exists $s_0 \in S$ with $\gamma < s_0 < \mu$. Let $k \geq r$ and let $m \in \omega$. Then $a_k(m) \in U(s_0, \beta, j) \subset W(s_0)$ so $W(a_k(m)) \subset W^2(s_0) \subset V(s_0)$. We show that $F(\beta, r) \subset V(s_0)$ so $V(s_0) \in \mathfrak{F}$ as required. Let $\langle k, n \rangle \in F(\beta, r)$. Then $k \geq r \geq j$ and $f_\beta(k) < n$ so

$$\langle k, n \rangle \in U(a_k(n), \beta, j) \subset W(a_k(n)) \subset V(s_0). \quad \blacksquare$$

COROLLARY. *The space X is not weakly orthocompact. (Indeed, X is a transitive space that is not preorthocompact; see [3, Lemma 6.16 and Corollary 5.11].)*

For the nonce, mimicking terminology of E. Hewitt, we say a topological space is q -complete provided it admits a complete quasi-uniformity. In this terminology, Proposition 3.12 of [3] shows that the perfect preimage of a q -complete space is q -complete. The present example shows that this result does not obtain if perfect maps are replaced by quasi-perfect maps.

3. Further properties of the example. X is almost finitely-fully normal. Let $\mathfrak{B}$ be an open cover of X . Without loss of generality, we assume that for each $x \in \mu$ there is $B_x \in \mathfrak{B}$ of the form $B_x = U(x, \beta_x, m_x)$ and that all remaining members of $\mathfrak{B}$ are isolated points. Since μ is a regular cardinal, by the Pressing-Down Lemma, there are a cofinal subset S of μ , $\beta \in \mu$, and $m \in \omega$ so that $\beta_x = \beta$ and $m_x = m$ for all $x \in S$.

Set $A = \{k \in \omega: \langle f_s(k) \rangle_{s \in S} \text{ is eventually bounded}\}$. By Lemma 2, A is a finite set. Let h be a natural number exceeding $\max(A)$. Set $R = (\beta, \mu) \cup \{\langle k, n \rangle: k \geq h, m \text{ and } f_\beta(k) < n\}$. Then R is an open set, and since $\mu \setminus R$ is compact there is a finite subset $\mathfrak{B}'$ of $\mathfrak{B}$ so that $\mu \setminus R \subset \bigcup \mathfrak{B}'$. Let

$$\mathfrak{R} = \mathfrak{B}' \cup \{R\} \cup \{\{x\}: x \notin \bigcup (\mathfrak{B}' \cup \{R\})\}.$$

Then $\mathfrak{R}$ is a locally finite open cover of X . Let M be a finite refiner of $\mathfrak{R}$. By Lemma 1 it suffices to show that M is a refiner of $\mathfrak{B}$. We assume $M \subset R$. Set $M_1 = M \cap \mu$ and set $M_2 = M \cap (\omega \times \omega)$. List the members of M_2 as $\langle k_1, n_1 \rangle, \langle k_2, n_2 \rangle, \dots, \langle k_j, n_j \rangle$, where $k_1 \leq k_2 \leq \dots \leq k_j$. Let $q = \max\{n_1, n_2, \dots, n_j\}$. There is an $s' \in S$ so that $s' > \max M_1$. Since $k_1 \geq h$, $k_1 \notin A$ so there is an $s \in S$ with $s > s'$ so $f_s(k_1) > q$. Let $1 \leq d \leq j$. Since $M \subset R$, $f_\beta(k_d) < n_d \leq q < f_s(k_1) \leq f_s(k_d)$. Thus $M_2 \subset \{\langle k, n \rangle: k \geq m, f_\beta(k) < n \leq f_s(k)\} \subset U(s, \beta, m)$. It follows that $M \subset U(s, \beta, m)$.

X is not almost $\aleph_0$ -fully normal. Set $\mathfrak{B}' = \{U(x, 0, 0): x \in \mu\}$ and set $\mathfrak{B} = \mathfrak{B}' \cup \{\{x\}: x \in \omega \times \omega\}$. Suppose $\mathcal{L}$ is an open locally finite cover of X with the property that every countable refiner of $\mathcal{L}$ is a refiner of $\mathfrak{B}$. Since $\mathcal{L}|_\mu$ is locally finite, and hence finite, only finitely many members of $\mathcal{L}$ meet μ ; list these members as $L_1, L_2, \dots, L_z$. For each $x \in \mu$, there are k_x, β_x and n_x so that $x \in U(x, \beta_x, n_x) \subset L_{k_x}$,

where $U(x, \beta_x, n_x)$ is a basic open set about x . By the Pressing-Down Lemma, there is a cofinal subset $\tilde{S}$ of μ , $\tilde{m} \in \omega$, $\tilde{k}$ with $1 \leq \tilde{k} \leq z$, and $\tilde{\beta} \in \mu$ so that for each $x \in \tilde{S}$, $k_x = \tilde{k}$, $\beta_x = \tilde{\beta}$ and $n_x = \tilde{m}$. Set $\tilde{A} = \{k \in \omega: \langle f_s(k) \rangle_{s \in \tilde{S}} \text{ is eventually bounded}\}$. Then $\tilde{A}$ is finite and there is a natural number $\tilde{h}$ exceeding $\max(\tilde{A})$. Set

$$\tilde{R} = (\tilde{\beta}, \mu) \cup \{\langle k, n \rangle: k \geq \tilde{h}, \tilde{m} \text{ and } f_{\tilde{\beta}}(k) < n\}.$$

Then $\tilde{R} \subset L_{\tilde{k}}$ so each countable subset of $\tilde{R}$ is a refiner of $\mathfrak{B}$. Let $D = \tilde{R} \setminus \mu$. Then D is a refiner of $\mathfrak{B}$ —a contradiction. ■

In his thesis [5, Theorem 2.2.10], H. J. K. Junnila proves that a space X is 2-fully normal if and only if it is almost 2-fully normal and for each open cover $\mathcal{C}$ of X there is a reflexive relation V on X such that, for each $x \in X$, $V(x)$ is open and such that, for each $x \in X$ and $y \in V(x)$, $V(x) \cup V(y)$ is a refiner of $\mathcal{C}$. In particular, every orthocompact almost 2-fully normal space is 2-fully normal. It is unknown whether the converse holds; the referee suggests that the space we have considered above could possibly be used in the construction of a counterexample.

Since the space X was constructed as an example of a space that does not have a countably-compact-ification, it is interesting to note that nearly the same method of proof establishes that the following countably compact normal space Y is not q -complete.

Let $\langle A_\alpha \rangle_{\alpha \in \mu}$ be an increasing maximal tower on ω , where μ is a regular cardinal. Let $Y = \mu \cup \omega$ and, as usual, define a topology on Y by specifying the following neighborhoods: Points of ω are isolated. If $0 \leq \beta < \alpha < \mu$ and F is a finite subset of ω , set

$$U(\alpha, \beta, F) = (\beta, \alpha] \cup [A_\alpha \setminus A_\beta] \setminus F,$$

and if $\alpha = 0$ and F is a finite subset of ω , set

$$U(0, \beta, F) = \{0\} \cup (A_0 \setminus F).$$

Then $\mathfrak{F} = \text{fil}\{(\omega \setminus A_\alpha) \setminus F: \alpha \in \mu \text{ and } F \text{ is a finite subset of } \omega\}$ is a filter without a cluster point that is a Cauchy filter with respect to each quasi-uniformity that Y admits.

The similarity of the methods of proof that X and Y are not q -complete is not just a coincidence. The basic neighborhoods of points of μ in X can be defined in terms of the following tower on $\omega \times \omega$: $\{\langle \langle k, n \rangle: k \in \omega, n \leq f_\alpha(k) \rangle\rangle_{\alpha \in \mu}$.

REFERENCES

1. G. Aquaro, *Intorno ad una generalizzazione degli spazi paracompatti*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **38** (1965), 824–827.
2. D. K. Burke and E. K. van Douwen, *On countably compact extensions of normal locally compact M -spaces*, Set-Theoretic Topology (Papers, Inst. Medicine and Math., Ohio Univ., Athens, Ohio, 1975–1976), Academic Press, New York, 1977, pp. 81–90.
3. P. Fletcher and W. F. Lindgren, *Quasi-uniform spaces*, Lecture Notes in Pure and Appl. Math., vol. 77, Dekker, New York, 1982.
4. K. P. Hart, *More on M. E. Rudin's Dowker space*, Proc. Amer. Math. Soc. **86** (1982), 508–510.

5. H. J. K. Junnila, *Covering properties and quasi-uniformities of topological spaces*, Ph. D. Thesis, Virginia Polytechnic Inst. and State Univ., Blacksburg, Va., 1978.
6. M. J. Mansfield, *Some generalizations of full normality*, Trans. Amer. Math. Soc. **86** (1957), 489–505.
7. K. Morita, *Paracompactness and product spaces*, Fund. Math. **50** (1961/1962), 223–236.
8. W. J. Pervin, *Quasi-uniformization of topological spaces*, Math. Ann. **147** (1962), 316–317.

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