FIBRATIONS AND HOMOTOPY TYPE

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ABSTRACT. We prove the results announced in [13]. These allow us to distinguish homotopy types of spherical fibrations using fibre homotopy invariants.

Introduction. Let E_1, E_2 be Hurewicz fibrations over a connected finite CW complex B . We say that they are *fibre homotopy equivalent* ($E_1 \simeq_B E_2$) when there are fibre preserving maps $\tau: E_1 \rightarrow E_2, \gamma: E_2 \rightarrow E_1$ such that $\gamma\tau, \tau\gamma$ are homotopic through fibre preserving maps to the identities on E_1, E_2 . When this happens τ defines a homotopy equivalence $(E_1, F_1) \simeq (E_2, F_2)$ of pairs, where F_1, F_2 are the fibres over a basepoint of B . A result of A. Dold [2] asserts that $E_1 \simeq_B E_2$ provided there is a fibre preserving map $\tau: E_1 \rightarrow E_2$ whose restriction to fibres is a homotopy equivalence $F_1 \simeq F_2$.

We take F_1, F_2 to be homotopy q -spheres S_1^q, S_2^q ($q \geq 1$), and we try to relate the fibre homotopy types of E_1, E_2 when either $E_1 \simeq E_2$ or $(E_1, S_1^q) \simeq (E_2, S_2^q)$. Results of I. M. James and J. H. C. Whitehead [7, 8] can be expressed in the following form.

THEOREM JW. *Let E_1, E_2 be orthogonal q -sphere bundles over a sphere S^{r+1} with $E_1 \simeq E_2$. If E_2 has a cross-section then*

- (i) E_1 has a cross-section, and
- (ii) *there is a homotopy equivalence $f: S^{r+1} \rightarrow S^{r+1}$ such that E_1, f^*E_2 have the same fibre homotopy type.*

Also, in [4], S. Y. Hussein proved

THEOREM H. *Let B be simply connected with $\dim B < q$ and suppose $(E_1, S_1^q) \simeq (E_2, S_2^q)$. Then there is a homotopy equivalence $f: B \rightarrow B$ such that E_1, f^*E_2 have the same fibre homotopy type.*

We wish to generalise these results to a situation where B is as general a space as possible and E_1, E_2 are not necessarily fibre bundles.

Chapters I and II are directed towards proving the following conjecture of I. M. James.

CONJECTURE J. *If $E_1 \simeq B \times S^q$ then E_1 is fibre homotopy trivial.*

We prove this when B is nilpotent [1] and either E_1 has a cross-section (I. Proposition 2) or $\dim B < 2q$ (II. Corollary 1). When B is a sphere or when $H^q(B; \mathbb{Z})$ is finite, Conjecture J is proved in [6], together with other special cases. We

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make repeated use of the following result, which can be proved using the counting argument of [6, §3].

THEOREM J. *If E_1, E_2 are $\mathbf{Z}$ -orientable with $E_1 \simeq E_2$, then if the Euler class $\chi(E_2)$ vanishes so does $\chi(E_1)$. (A similar result holds for the mod p Euler classes, and also for the localizations of the fibrations E_1, E_2 .)*

Although we use counting arguments in Chapter I, in the spirit of [6 and 7], there are several new ingredients in our proofs. The first is the use of localization (suggested to me by Professor James in connection with Conjecture J) in Chapters I and II. The second is the notion of *an exchange* which appears in Chapter II. When a homotopy equivalence $\tau: E_1 \rightarrow E_2$ looks like a fibre homotopy equivalence from the point of view of cohomology, there is *no exchange*; otherwise we say there is an exchange. The key point in the proof of II. Theorem 1 is that when there is an exchange B is describable in terms of $E_2|B^{q-1}$. Another novelty, in Chapter III, is the use of a co- H -spacelike structure on a suspended fibration to replace a homotopy equivalence by a fibre homotopy equivalence. This replaces Husseini's loop space argument and proves a stronger result.

Our results apply mainly in situations where both E_1 and E_2 have cross-sections (but see II. Theorem 1 for an important exception). There are many unanswered questions when either E_1 or E_2 does not have a cross-section, and for background we refer to [9]. For example, does the following conjecture hold?

CONJECTURE N. If $E_1 \simeq E_2$, where E_2 has a cross-section, then E_1 has a cross-section.

According to Proposition 2 this would imply Conjecture J when B is nilpotent.

Finally, a word of thanks: the present paper is a combination of three unpublished notes (corresponding to the three chapters), and the presentation has been substantially improved by the helpful comments of two anonymous referees. Also my original proof of III. Theorem 2(i) required E_2 to be $\mathbf{Z}$ -orientable and B to be nilpotent. I owe to a referee the observation that these hypotheses are unnecessary.

CHAPTER I. COUNTING ARGUMENTS

1. Statement of results. Let p be a prime. When B is nilpotent and $E_j \rightarrow B$ is $\mathbf{Z}$ -orientable ($j = 1, 2$), we can form the localization $E_{j(p)} \rightarrow B_{(p)}$. We denote the projections of $E_j, E_{j(p)}$ by π_j .

PROPOSITION 1. *Let B be nilpotent, and let E_2 be $\mathbf{Z}$ -orientable. Suppose, for some $1 \leq i \leq q$, $E_{2(2)}|(B^{q+i})_{(2)}$ has a cross-section and $E_1 \simeq E_2$. If the i th Stiefel-Whitney class $w_i(E_2) \in H^i(B; \mathbf{Z}/(2))$ vanishes then so does $w_i(E_1)$.*

REMARKS. (i) This is proved by an extension of the method used to prove Theorem J. If E_1 is known to be $\mathbf{Z}$ -orientable then it suffices to assume $E_{1(2)} \simeq E_{2(2)}$ instead of $E_1 \simeq E_2$.

(ii) A similar result holds at the odd primes. It is necessary to know that E_1 is $\mathbf{Z}$ -orientable and to replace the Stiefel-Whitney classes by the Wu classes [11, p. 229].

(iii) If $f: B \rightarrow B$ is a homotopy equivalence, then $(E_2, S_2^q) \simeq (f^*E_2, S_2^q)$, but $w_i(E_2) \neq w_i(f^*E_2)$ in general. For example, we may take $f: S^2 \times S^2 \rightarrow S^2 \times S^2$ to be the switching map, and E_2 to be induced from the Hopf circle bundle over S^2 by projection to the second factor. We may fibre suspend twice to get an example with $\dim B < q$.

PROPOSITION 2. *Let B be nilpotent and suppose $E_1 \simeq B \times S^q$. If $E_{1(p)}$ has a cross-section for all primes p then E_1 is fibre homotopy trivial.*

This bears on Conjecture J. We have more to say in Chapter II.

EXAMPLES. (i) Let E_1 be the fibre suspension of the tangent sphere bundle of the complex projective plane CP^2 , and let E_2 be the fibre suspension of the fibre join of two copies of the Hopf circle bundle over CP^2 . Because both bundles have cross-sections their cohomology rings are isomorphic. However $E_1 \not\simeq E_2$. In effect Proposition 1 says that the homotopy types of E_1, E_2 are distinguished by the operation Sq^2 .

(ii) Suppose $E_1 \simeq B \times S^q$ with $\dim B \leq q + 1$. By Proposition 1, E_1 is $\mathbf{Z}$ -orientable, and so E_1 has a cross-section by Theorem J. If B is nilpotent then E_1 is fibre homotopy trivial by Proposition 2. This proves Conjecture J when $\dim B \leq q + 1$ and B is nilpotent, but we do better in Chapter II.

(iii) There are sphere bundles $E_1 \not\simeq E_2$ over a sphere, and a finite CW complex A , such that $A \times E_1 \simeq A \times E_2$ [3]. Proposition 2 with [6, 2.2] shows that this cannot happen when E_2 is trivial.

2. Characteristic classes. In the context of Proposition 1 let s_2' be a cross-section of $E_{2(2)}|(B^{q+i})_{(2)}$ and let π_2' be the restriction of π_2 to $E_{2(2)}|(B^{q+i})_{(2)}$. We denote the inclusions $(B^{q+i})_{(2)} \rightarrow B_{(2)}$, $E_{2(2)}|(B^{q+i})_{(2)} \rightarrow E_{2(2)}$ by ι' . Then the $\mathbf{Z}/(2)$ -Gysin sequences give us the following commuting diagrams with exact rows ($r \geq 0$).

$$\begin{array}{ccccccc} 0 & \rightarrow & H^r B_{(2)} & \xrightarrow{\pi_2^*} & H^r E_{2(2)} & \xrightarrow{\psi_2} & H^{r-q} B_{(2)} \rightarrow 0 \\ & & \downarrow \iota'^* & & \downarrow \iota'^* & & \downarrow \iota'^* \\ 0 & \rightarrow & H^r (B^{q+i})_{(2)} & \xrightarrow{\pi_2'^*} & H^r E_{2(2)}|(B^{q+i})_{(2)} & \xrightarrow{\psi_2'} & H^{r-q} (B^{q+i})_{(2)} \rightarrow 0 \end{array}$$

Let $1 \in H^0(B^{q+i})_{(2)}$ be the generator and let $a_0 \in H^q(E_{2(2)}|(B^{q+i})_{(2)})$ satisfy $\psi_2' a_0 = 1$. If $a_2' = a_0 - \pi_2'^* s_2'^* a_0$ then $\psi_2' a_2' = 1$ and $s_2'^* a_2' = 0$.

Now $\iota'^*: H^r E_{2(2)} \rightarrow H^r(E_{2(2)}|(B^{q+i})_{(2)})$ is surjective for $r < q + i$, and we let $a_2 \in H^q E_{2(2)}$ satisfy $\iota'^* a_2 = a_2'$. Then $\psi_2 a_2 = 1$ so $\psi_2 Sq^i a_2$ is the localization of $w_i(E_2)$. But $w_i(E_2) = 0$ by hypothesis, so $Sq^i a_2 = \pi_2'^* b$ for some $b \in H^{q+i} B_{(2)}$. Therefore

$$Sq^i a_2' = \iota'^* Sq^i a_2 = \iota'^* \pi_2'^* b = \pi_2'^* \iota'^* b,$$

so we have

$$\iota'^* b = s_2'^* \pi_2'^* \iota'^* b = s_2'^* Sq^i a_2' = Sq^i s_2'^* a_2' = 0.$$

Therefore $b = 0$ because $\iota'^*: H^r B_{(2)} \rightarrow H^r(B^{q+i})_{(2)}$ is injective for $r \leq q + i$. It follows that $Sq^i a_2 = 0$ so

$$Sq^i(H^q E_{2(2)}) = \pi_2'^* Sq^i(H^q B_{(2)}).$$

For a general CW complex X let $\#(i)X$ be the order of the subgroup $\text{Sq}^i(H^q X)$ of $H^{q+i}X$. We note that if X is nilpotent then $\#(i)X = \#(i)X_{(2)}$. Then $\#(i)E_2$, $\#(i)B < \infty$, and we know that $\#(i)E_2 = \#(i)E_{2(2)} = \#(i)B_{(2)} = \#(i)B$. By Theorem J, $w_{q+1}(E_1) = 0$ because $w_{q+1}(E_2) = 0$, so the $\mathbf{Z}/(2) = \text{Gysin}$ sequence of E_1 is

$$0 \rightarrow H^r B \xrightarrow{\pi_1^*} H^r E_1 \xrightarrow{\psi_1} H^{r-q} B \rightarrow 0$$

for $r \geq 0$. Let $a_1 \in H^q E_1$ satisfy $\psi_1 a_1 \neq 0$. Then $w_i(E_1) = \psi_1 \text{Sq}^i a_1 = 0$ since $\#(i)E_1 = \#(i)E_2 = \#(i)B$. This proves Proposition 1.

3. Decomposability. If $E_1 \simeq B \times S^q$ then, by Proposition 1, E_1 is $\mathbf{Z}$ -orientable. We observe that if $E_{1(p)}$ is fibre homotopy trivial for all primes p then E_1 is fibre homotopy trivial. One way of seeing this is to note that a fibre homotopy equivalence corresponds as in [14, §1] to a cross-section of a function space fibration. We may then use [15, 3.2].

We denote the inclusion of the fibre in $E_{1(p)}$ by ι_1 . Then we have an exact sequence

$$0 \rightarrow H^q B_{(p)} \xrightarrow{\pi_1^*} H^q E_{1(p)} \xrightarrow{\iota_1^*} H^q S_{(p)}^q \rightarrow 0$$

with coefficients $\mathbf{Z}/(p)$. For a general CW complex X we call $x \in H^q X$ *cospherical* when it lies in the image of $f^*: H^q S_{(p)}^q \rightarrow H^q X$ for some $f: X \rightarrow S_{(p)}^q$. Let $\#X$ be the number of cospherical elements of $H^q X$.

Then $\#E_{1(p)}$, $\#B_{(p)} < \infty$ and $\#E_{1(p)} = \#(B \times S^q)_{(p)} \geq (\#B_{(p)}) + 1$. So some cospherical $a \in H^q E_{1(p)}$ is not $\pi_1^* c$ for any cospherical $c \in H^q B_{(p)}$. Now $\iota_1^* a \neq 0$ because otherwise we would have $a = \pi_1^* b$, and b would have to be cospherical (it is here that we use the hypothesis that $E_{1(p)}$ has a cross-section). So for some $f: E_{1(p)} \rightarrow S_{(p)}^q$ the composite $f\iota_1: S_{1(p)}^q \rightarrow S_{(p)}^q$ is a homotopy equivalence (here we see the usefulness of localizing at p). So $E_{1(p)}$ is fibre homotopy trivial, and this proves Proposition 2.

CHAPTER II. CROSS-SECTIONS

1. Statement of results. The proof of Theorem JW(i) uses a counting argument and the key point is that $G = \pi_* S_1^q$ satisfies the *endomorphism condition*, namely the surjective endomorphisms of G are isomorphisms. It suffices for G to be finitely generated, and so in [7] the fibres of E_1, E_2 are not required to be spheres.

In our situation B is no longer a suspended space, and the counting argument of [7] is not available. However, taking p to be a prime, we have the following p -local result.

THEOREM 1. *Let B be nilpotent, and let E_1, E_2 be $\mathbf{Z}$ -orientable with $\dim B < 2q$. Suppose E_2 has a cross-section and $E_{1(p)} \simeq E_{2(p)}$ where p is prime. Then $E_{1(p)}$ has a cross-section.*

Comparing this with I. Proposition 2 we obtain

COROLLARY 1. *Let B be nilpotent with $\dim B < 2q$. If $E_1 \simeq B \times S^q$ then E_1 is fibre homotopy trivial.*

So we have another special case of Conjecture J. Theorem 1 appears to support Conjecture N, although it seems to be unknown whether E_1 has a cross-section when $E_{1(p)}$ has a cross-section for all primes p . (There is no difficulty when B is a sphere, or when $H^q(B; Q) = 0$, or, less trivially, when q is odd.) Interestingly, the endomorphism condition for finitely generated $\mathbf{Z}_{(p)}$ -modules plays a part in our proof of Theorem 1.

2. No exchange. Let s_2 be a cross-section of $E_{2(p)}$ and let $\tau: E_{1(p)} \rightarrow E_{2(p)}$ be a homotopy equivalence with homotopy inverse γ . By Theorem J we have $\chi(E_{1(p)}) = 0$ because $\chi(E_{2(p)}) = 0$. This proves Theorem 1 when $q = 1$, and we exclude this case for the remainder of Chapter II. The Gysin sequences give us the diagrams

$$\begin{array}{ccccccc} 0 & \rightarrow & H^r B_{(p)} & \xrightarrow{\pi_1^*} & H^r E_1(p) & \xrightarrow{\psi_1} & H^{r-q} B_{(p)} \rightarrow 0 \\ & & & & \gamma^* \downarrow \uparrow \tau^* & & \\ 0 & \rightarrow & H^r B_{(p)} & \xrightarrow{\pi_2^*} & H^r E_{2(p)} & \xrightarrow{\psi_2} & H^{r-q} B_{(p)} \rightarrow 0 \end{array}$$

with exact rows ($r \geq 0$). Here we take coefficients in the ring $\mathbf{Z}_{(p)}$ of integers localized at p . Proceeding as in I, §2, let $1 \in H^0 B_{(p)}$ be the identity and let $a_0 \in H^q E_{2(p)}$ satisfy $\psi_2 a_0 = 1$. Let $a_2 = a_0 - \pi_2^* s_2^* a_0$. Then $\psi_2 a_2 = 1$ and $s_2^* a_2 = 0$. We say there is *no exchange* when $\psi_1 \tau^* a_2$ is a unit of $H^0 B_{(p)}$. For the remainder of §2 we assume this is the case, and we let $a_1 = \tau^* a_2$.

We note that, for $r < q$, $s_2^* \gamma^* \pi_1^*: H^r B_{(p)} \rightarrow H^r B_{(p)}$ is a $\mathbf{Z}_{(p)}$ -module automorphism. For $r \geq q$ let $b \in H^r B_{(p)}$ and, as in [10], write $\tau^* \pi_2^* b$ in the form

$$\pi_1^* c + a_1 \cdot \pi_1^* d,$$

where $c \in H^r B_{(p)}$, $d \in H^{r-q} B_{(p)}$. Then b is

$$s_2^* \gamma^* (\pi_1^* c + a_1 \cdot \pi_1^* d) = s_2^* \gamma^* \pi_1^* c + (s_2^* a_2) \cdot (s_2^* \gamma^* \pi_1^* d).$$

Since $s_2^* a_2 = 0$, we have $b = s_2^* \gamma^* \pi_1^* c$, so the $\mathbf{Z}_{(p)}$ -module endomorphism $s_2^* \gamma^* \pi_1^*$ of $H^r B_{(p)}$ is surjective. But $H^r B_{(p)}$ is a finitely generated $\mathbf{Z}_{(p)}$ -module, so $s_2^* \gamma^* \pi_1^*$ is an automorphism for all r .

So $\pi_1 \gamma s_2: B_{(p)} \rightarrow B_{(p)}$ is a homotopy equivalence. Let β be a homotopy inverse. Then $\gamma s_2 \beta$ is homotopic to a cross-section of $E_{1(p)}$, and this proves Theorem 1 when there is no exchange.

3. Exchange. We treat the case where $\psi_1 \tau^* a_2$ is not a unit in $\mathbf{Z}_{(p)}$. We say there is *an exchange*, and it is here that we use the hypothesis $\dim B < 2q$. We denote the

inclusions $(B^{q-1})_{(p)} \rightarrow B_{(p)}$, $E_{2(p)}|(B^{q-1})_{(p)} \rightarrow E_{2(p)}$ by ι . The Gysin sequences give us the commuting diagrams

$$\begin{array}{ccccccc}
 0 & \rightarrow & H^r B_{(p)} & \xrightarrow{\pi_1^*} & H^r E_{1(p)} & \xrightarrow{\psi_1} & H^{r-q} B_{(p)} \rightarrow 0 \\
 & & & & \gamma^* \downarrow \uparrow \tau^* & & \\
 0 & \rightarrow & H^r B_{(p)} & \xrightarrow{\pi_2^*} & H^r E_{2(p)} & \xrightarrow{\psi_2} & H^{r-q} B_{(p)} \rightarrow 0 \\
 & & \downarrow \iota^* & & \downarrow \iota^* & & \downarrow \iota^* \\
 0 & \rightarrow & H^r (B^{q-1})_{(p)} & \xrightarrow{\pi_2'^*} & H^r E_{2(p)}|(B^{q-1})_{(p)} & \xrightarrow{\psi_2'} & H^{r-q} (B^{q-1})_{(p)} \rightarrow 0
 \end{array}$$

with exact rows ($r \geq 0$). Here π_2' denotes the restriction of π_2 , and we take coefficients in either of the rings $\mathbf{Z}_{(p)}$ or $\mathbf{Z}_{(p)}/(p) \cong \mathbf{Z}/(p)$.

With coefficients $\mathbf{Z}_{(p)}/(p)$ we have $\psi_1 \tau^* a_2 = 0$ so $\tau^* a_2 = \pi_1^* b_1$ for some $b_1 \in H^q B_{(p)}$. Let $\mathfrak{B}$ be the Bockstein associated with the exact sequence

$$0 \rightarrow \mathbf{Z}_{(p)} \xrightarrow{p} \mathbf{Z}_{(p)} \rightarrow \mathbf{Z}_{(p)}/(p) \rightarrow 0.$$

Then $\mathfrak{B} a_2 = 0$, so $0 = \tau^* \mathfrak{B} a_2 = \mathfrak{B} \tau^* a_2 = \mathfrak{B} \pi_1^* b_1 = \pi_1^* \mathfrak{B} b_1$. Therefore $\mathfrak{B} b_1 = 0$ so b_1 is the reduction of a $\mathbf{Z}_{(p)}$ -cohomology class. Hence,

LEMMA 1. *The homomorphism*

$$\iota^* \gamma^* \pi_1^*: H^q(B_{(p)}; \mathbf{Z}_{(p)}) \rightarrow H^q(E_{2(p)}|(B^{q-1})_{(p)}; \mathbf{Z}_{(p)})$$

is surjective.

Working with coefficients $\mathbf{Z}_{(p)}/(p)$, we suppose $\iota^* \gamma^* \pi_1^* b = 0$ where $b \in H^r B_{(p)}$ and $r \geq q$. Since $\dim B < 2q$, either $b = 0$ or $r < 2q$. If $r < 2q$ then $\iota^*: H^{r-q} B_{(p)} \rightarrow H^{r-q} (B^{q-1})_{(p)}$ is injective. But $0 = \psi_2' \iota^* \gamma^* \pi_1^* b = \iota^* \psi_2 \gamma^* \pi_1^* b$, so $\gamma^* \pi_1^* b = \pi_2^* c$ for some $c \in H^r B_{(p)}$. Now $\tau^*(a_2 \cdot \pi_2^* c) = (\pi_1^* b_1)(\pi_1^* b) = 0$ since $\dim B < 2q$ so $a_2 \cdot \pi_2^* c = 0$. Applying ψ_2 to this equation we find that $c = 0$. So $\gamma^* \pi_1^* b = 0$ and $b = 0$. This proves

LEMMA 2. *For $r \geq q$ the homomorphism $\iota^* \gamma^* \pi_1^*: H^r B_{(p)} \rightarrow H^r(E_{2(p)}|(B^{q-1})_{(p)})$ (with coefficients $\mathbf{Z}_{(p)}/(p)$) is injective.*

Still working with coefficients $\mathbf{Z}_{(p)}/(p)$, we observe that since $\psi_1 \tau^* a_2 = 0$ we have $\psi_1 \tau^* \pi_2^* b_2 \neq 0$ for some $b_2 \in H^q B_{(p)}$. Let $a_1 = \tau^* \pi_2^* b_2$. If $x \in H^r E_{2(p)}$, where $q \leq r < 2q - 1$, we write $\tau^* x$ in the form $\pi_1^* c + a_1 \cdot \pi_1^* d$, where $c \in H^r B_{(p)}$, $d \in H^{r-q} B_{(p)}$. Then

$$\iota^* x = \iota^* \gamma^* (\pi_1^* c + a_1 \cdot \pi_1^* d),$$

which we write as

$$\iota^* \gamma^* \pi_1^* c + \iota^* ((\pi_2^* b_2) \cdot (\gamma^* \pi_1^* d)).$$

Since $r < 2q - 1 < 2q$, we have $\gamma^* \pi_1^* d = \pi_2^* e$ for some $e \in H^{r-q} B_{(p)}$. So

$$\iota^* x = \iota^* \gamma^* \pi_1^* c + \iota^* (\pi_2^* (b_2 \cdot e)) = \iota^* \gamma^* \pi_1^* c + \pi_2'^* \iota^* (b_2 \cdot e) = \iota^* \gamma^* \pi_1^* c$$

because $H^r(B^{q-1})_{(p)} = 0$. But $\iota^*: H^r E_{2(p)} \rightarrow H^r E_{2(p)} | (B^{q-1})_{(p)}$ is surjective for $q \leq r < 2q - 1$, and this proves

LEMMA 3. For $q \leq r < 2q - 1$ the homomorphism

$$\iota^* \gamma^* \pi_1^*: H^r B_{(p)} \rightarrow H^r (E_{2(p)} | (B^{q-1})_{(p)})$$

(with coefficients $\mathbf{Z}_{(p)}/(p)$) is surjective.

We form the induced fibrations $\pi_1^* E_{1(p)}$, $E = \iota^* \gamma^* \pi_1^* E_{1(p)}$ over $E_{1(p)}$, $E_{2(p)} | (B^{q-1})_{(p)}$.

$$\begin{array}{ccccc} E & \rightarrow & \pi_1^* E_{1(p)} & \rightarrow & E_{1(p)} \\ s \uparrow \downarrow & & \downarrow & & \downarrow \\ E_{2(p)} | (B^{q-1})_{(p)} & \xrightarrow{\gamma^*} & E_{1(p)} & \xrightarrow{\pi_1} & B_{(p)} \end{array}$$

Then $\pi_1^* E_{1(p)}$ has a cross-section so E has a cross-section s . But $\chi(E_{1(p)}) = 0$, and obstruction theory with Lemmas 1–3 shows that s gives rise to a cross-section of $E_{1(p)}$. This completes the proof of Theorem 1.

CHAPTER III. RELATIVE HOMOTOPY TYPES

1. Statement of results. In this chapter we are concerned with the situation when E_2 is a fibre suspension ΣE for some $E \rightarrow B$. This implies E_2 has a cross-section, and some other remarks about this condition are made in §2.

THEOREM 2. Suppose E_2 is a fibre suspension and either

(i) $(E_1, S_1^q) \simeq (E_2, S_2^q)$, or

(ii) $E_1 \simeq E_2$ with $H^q(B; \mathbf{Z})$ finite, E_2 $\mathbf{Z}$ -orientable, and B nilpotent.

Then there is a homotopy equivalence $f: B \rightarrow B$ such that $E_1, f^* E_2$ have the same fibre homotopy type.

REMARKS. (i) We see in §2 that if $\dim B < q$, then E_2 is automatically a fibre suspension. Consequently, Theorem 2 improves Theorem H. Note that [4, Theorem 1.1] is incorrectly stated: this is shown by I, §1, Remark (iii).

(ii) Taking $B = S^{r+1}$ in Theorem 2 we obtain Theorem JW except for the case where $r = q - 1$. We deal with this case using our methods in the Appendix. Because we treat general spherical fibrations, and not just orthogonal sphere bundles, we can add to the picture in [8] with the following result.

COROLLARY 2. Let $\Phi \subseteq \pi_{r+q} S^q$ ($0 < r < q - 1$) be maximal with respect to the property that if $\alpha, \beta \in \Phi$ satisfy $\alpha + \beta = 0$ then $\alpha = \beta$. Let ϕ be the evident function from Φ onto the set of homotopy types of q -spherical fibrations over S^{r+1} . Then ϕ is a bijection.

EXAMPLE. Let α generate $\pi_8 S^5 \cong \mathbf{Z}/(24)$ and let E_1, E_2 be the fibrations with fibre S^5 over S^4 classified by $5\alpha, 11\alpha$. Then for each prime p there is a homotopy equivalence $f: S_{(p)}^4 \rightarrow S_{(p)}^4$ such that $E_{1(p)}, f^* E_{2(p)}$ have the same fibre homotopy type. In particular, E_1, E_2 have the same genus [12]. But $E_1 \not\simeq E_2$.

2. Desuspension. Let G_1^q be the H -space of maps $S^q \rightarrow S^q$ of degree 1 with multiplication given by composition. Then, according to [17], there is a q -spherical fibration U^q over the classifying space BG_1^q with the following property. For any CW complex X the correspondence $f \rightarrow f^*U^q$ defines a bijection from homotopy classes of maps $X \rightarrow BG_1^q$ to degree 1 fibre homotopy types of q -spherical fibrations over X . In particular, the fibre suspension ΣU^{q-1} is induced by a map $g: BG_1^{q-1} \rightarrow BG_1^q$, and degree 1 fibre homotopy classes of $q-1$ spherical fibrations over X which fibre suspend to f^*U^q correspond bijectively to factorisations up to homotopy of f through g . But a calculation using [18] shows that $g_*: \pi_r BG_1^{q-1} \rightarrow \pi_r BG_1^q$ is an isomorphism for $r < q-1$ and an epimorphism for $r = q-1$. This proves

PROPOSITION 3. *If $\dim B < q$, then E_2 is fibre homotopy equivalent to the fibre suspension of a $(q-1)$ -spherical fibration E .*

We next prove

PROPOSITION 4. *If E_2 is a q -sphere bundle (not necessarily orthogonal) with a cross-section, then $E_2 \simeq_B \Sigma E$ for some bundle E .*

We choose s_2 to map a basepoint of B to $(0, \dots, 1)$ in the fibre S_2^q of E_2 . The proof of [16, Lemma 1] shows that $E'_2 = E_2 - s_2(B)$ is a fibre bundle. Since the fibre of E'_2 is contractible, there is a cross-section s'_2 of E'_2 . We choose s'_2 to map the basepoint to $(0, \dots, -1) \in S_2^q$. Then again, $E = E'_2 - s'_2(B)$ is a fibre bundle.

We define the fibre suspension ΣE in the usual way as a quotient of $E \times [-1, 1]$. Now define $T: s_2(B) \cup s'_2(B) \cup S_2^q \rightarrow [-1, 1]$ by $T(e) = 1, -1$, or x_q according as $e \in s_2(B)$, $e \in s'_2(B)$, or $e = (x_0, \dots, x_q) \in S_2^q$. Extending T to a map $T: E_2 \rightarrow [-1, 1]$, we define a fibre preserving map $h: E_2 \rightarrow \Sigma E$ by $h(e) = [e, T_2(e)]$ for $e \in E$. We define $h(e)$ to be $[e', -1]$ for $e \in s_2(B)$, $e \in s'_2(B)$, where $\pi_2(e') = \pi_2(e)$ and $e' \in E$. Then it is easy to check that h restricts to a homotopy equivalence on fibres. This proves Proposition 4.

3. No exchange. We argue from the situation of Theorem 2, except that we do not yet require E_2 to be a fibre suspension: it suffices to suppose that E_2 has a cross-section s_2 . Let $\tau: E_1 \rightarrow E_2$ be a homotopy equivalence with homotopy inverse γ (in the case of Theorem 2(ii) we take τ, γ to be homotopy equivalences of pairs). Let $\iota_j: S_j^q \rightarrow E_j$ be the inclusions.

LEMMA. *The composite $\pi_1 \gamma s_2: B \rightarrow B$ is a homotopy equivalence.*

To prove this in the case of Theorem 2(i) we have the commuting diagrams ($r \geq 0$)

$$\begin{array}{ccc} \pi_r(E_2, S_2^q) & \xrightarrow{\gamma} & \pi_r(E_1, S_1^q) \\ \pi_{2*} \downarrow \uparrow s_{2*} & & \downarrow \pi_{1*} \\ \pi_r(B, \cdot) & \xrightarrow{(\pi_1 \gamma s_2)_*} & \pi_r(B, \cdot) \end{array}$$

from which it follows that the $(\pi_1 \gamma s_2)_*$ are isomorphisms for all r . In Case (ii) we refer to the argument of II, §2, and this completes the proof.

We now require E_2 to be ΣE and let s_2 be the cross-section given by $s_2(b) = [e, 1]$ where $\pi_2(e) = b$. We define a fibre preserving map $\rho: \Sigma E \rightarrow \Sigma E$ by $\rho[e, t] = [e, -t]$. Then ρs_2 is another cross-section of E_2 , and $\pi_1 \gamma \rho s_2$ is a homotopy equivalence by the preceding lemma. Let β be a homotopy inverse. Then $\gamma \rho s_2 \beta$ is homotopic to a cross-section s_1 of E_1 and $s_1(\pi_1 \gamma \rho s_2) \simeq \gamma \rho s_2$. We note that s_2 is a cofibration, and so we can choose $\gamma' \simeq \gamma \rho$ so that the following diagram commutes.

$$\begin{array}{ccc} B & \xrightarrow{\pi_1 \gamma \rho s_2} & B \\ s_2 \downarrow & & \downarrow s_1 \\ \Sigma E & \xrightarrow{\gamma'} & E_1 \end{array}$$

Given maps $\theta_1, \theta_2, \theta_3: \Sigma E \rightarrow X$ satisfying $\theta_1 s_2 = \theta_2 \rho s_2$, $\theta_2 s_2 = \theta_3 \rho s_2$, we define a map $\ast(\theta_1, \theta_2, \theta_3): \Sigma E \rightarrow X$ by

$$\begin{aligned} [e, t] &\mapsto \theta_1[e, 3t + 2], & -1 \leq t \leq -1/3, \\ &\mapsto \theta_2[e, 3t], & -1/3 \leq t \leq 1/3, \\ &\mapsto \theta_3[e, 3t - 2], & 1/3 \leq t \leq 1. \end{aligned}$$

We choose $X = E_1$ and $\theta_1 = s_1 \pi_1 \gamma'$, $\theta_2 = s_1 \pi_1 \gamma \rho s_2 \pi_2$, $\theta_3 = \gamma' \rho$. Then the necessary equations are satisfied, and we form $\theta = \ast(\theta_1, \theta_2, \theta_3): \Sigma E \rightarrow E_1$. Now $\pi_1 \theta = \ast(\pi_1 \gamma', \pi_1 \gamma \rho s_2 \pi_2, \pi_1 \gamma' \rho): \Sigma E \rightarrow B$ and we note that the restriction of $\pi_1 \gamma \rho s_2 \pi_2$ to the fibre over any given point $b \in B$ is constant.

Consequently $\pi_1 \theta$ is homotopic to $\Sigma E \rightarrow B$ given by

$$\begin{aligned} [e, t] &\mapsto \pi_1 \gamma'[e, 2t + 1], & -1 \leq t \leq 0, \\ &\mapsto \pi_1 \gamma'[e, 1 - 2t], & 0 \leq t \leq 1. \end{aligned}$$

But this is homotopic to $\Sigma E \rightarrow B$ given by

$$[e, t] \mapsto \pi_1 \gamma'[e, -1],$$

which is $\pi_1 \gamma' \rho s_2 \pi$. But $\pi_1 \gamma' \rho s_2 \pi_2 \simeq \pi_1 \gamma s_2 \pi_2$.

Since E_1 is a fibration we can choose $\theta' \simeq \theta$ so that the diagram

$$\begin{array}{ccc} \Sigma E & \xrightarrow{\theta'} & E_1 \\ \pi_2 \downarrow & & \downarrow \pi_1 \\ B & \xrightarrow{\pi_1 \gamma s_2} & B \end{array}$$

commutes. Applying our lemma again, we see that $\pi_1 \gamma s_2$ is a homotopy equivalence.

4. Fibres. To prove Theorem 2 it suffices to show that the restriction $\phi': S_2^q \rightarrow S_1^q$ of θ' to fibres is a homotopy equivalence. In either case we note that

$$\iota_1 \phi' = \theta' \iota_2 \simeq \theta \iota_2 = * (s_1 \pi_1 \gamma', s_1 \pi_1 \gamma \rho s_2 \pi_2, \gamma' \rho) \iota_2$$

and $s_1 \pi_1 \gamma \rho s_2 \pi_2 \iota_2$ is constant.

Case (i). The homomorphism $(\iota_1 \phi')_*: \pi_q S_2^q \rightarrow \pi_q E_1$ is

$$(s_1 \pi_1 \gamma' \iota_2)_* + (\gamma' \rho \iota_2)_* = (s_1 \pi_1 \gamma \rho \iota_2)_* + (\gamma \rho^2 \iota_2)_*,$$

which is $(\gamma \iota_2)_*$. Now γ is a $\phi'': S_2^q \rightarrow S_1^q$. But the counting argument of [6, §2] proves that $\iota_{1*}: \pi_q S_1^q \rightarrow \pi_q E_1$ is injective so $\phi' \simeq \phi''$.

Case (ii). By Proposition 1 E_1 is $\mathbf{Z}$ -orientable because E is, and by Theorem J $\chi(E_1)$ vanishes because $\chi(E_2)$ does. So we have the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & H^q B & \xrightarrow{\pi_1^*} & H^q E_1 & \xrightarrow{\iota_1^*} & H^q S_1^q \rightarrow 0 \\ & & & & \gamma^* \downarrow \uparrow \tau^* & & \\ 0 & \rightarrow & H^q B & \xrightarrow{\pi_2^*} & H^q E_2 & \xrightarrow{\iota_2^*} & H^q S_2^q \rightarrow 0 \end{array}$$

with exact rows. Here we take coefficients in $\mathbf{Z}$. Now

$$\iota_1 \phi' = \phi' \iota_2 \simeq \theta \iota_2 = * (s_1 \pi_1 \gamma', s_1 \pi_1 \gamma \rho s_2 \pi_2, \gamma' \rho) \iota_2$$

and we note that $s_1 \pi_1 \gamma \rho s_2 \pi_2 \iota_2$ is constant. So $(\iota_1 \phi')^*: H^q E_1 \rightarrow H^q S_2^q$ is $(s_1 \pi_1 \gamma' \iota_2)^* + (\gamma' \rho \iota_2)^*$. Since $H^q(B; \mathbf{Z})$ is finite we must have $(\pi_1 \gamma' \iota_2)^*$ trivial, so $(\iota_1 \phi')^* = (\gamma' \rho \iota_2)^* = (\gamma \iota_2)^* = \iota_2^* \gamma^*$, which is surjective. So ϕ' is a homotopy equivalence and this completes the proof.

APPENDIX. $B = S^q$

Let $B = S^q$ ($q > 1$) and let $\tau: E_1 \rightarrow E_2$ be a homotopy equivalence with homotopy inverse γ . Let ξ, ξ_2 generate $H^q S^q, H^q S_2^q$ with coefficients in $\mathbf{Z}$. Let $(\pi_1 \gamma \iota_2)^* \xi = m \xi_2$. We say there is *no exchange* at the set M of primes which divide m . Then if s_2 is a cross-section of the localized fibration E_{2M} , $(\pi_1 \gamma)_M s_2: S_M^q \rightarrow S_M^q$ is a homotopy equivalence. This follows from the argument of II, §2.

A calculation, using [5, Theorem 7.8] for example, shows that E_2 is fibre homotopy equivalent to ΣE for some $(q-1)$ -spherical fibration E . We identify E_{2M} with $\Sigma(E_M)$ and let s_2 be the cross-section given by $s_2(b) = [e, 1]$ where $\pi_{2M}(e) = b$. As in III, §3 we obtain a commuting diagram

$$\begin{array}{ccc} \Sigma(E_M) & \xrightarrow{\theta} & E_{1M} \\ \pi_{2M} \downarrow & & \downarrow \pi_{1M} \\ B_M & \xrightarrow{(\pi_1 \gamma)_M s_2} & B_M. \end{array}$$

Let $\phi': S_{2M}^q \rightarrow S_{1M}^q$ be the restriction of θ' to fibres. As in III, §4 we see that $(\iota_{1M} \phi')^*: H^q E_{1M} \rightarrow H^q S_{2M}^q$ is $(\gamma \iota_2)_M^* + (s_1(\pi_1 \gamma \iota_2)_M)^*$, where s_1 is a cross-section of E_{1M} . Because of the way we chose M , $(\pi_1 \gamma \iota_2)_M^*$ is not surjective, so $\phi'^* \iota_{1M}^*$ is

surjective. Therefore $(\iota_{1M}\phi')^* = \phi'^*\iota_{1M}^*$ is surjective, and ϕ' is a homotopy equivalence.

Let N be the set of primes which do not divide m . We say there is an exchange at N . Note that E_{1N}, E_{2N} are fibre homotopy trivial. Since $\pi_{q-1}\text{SO}(q+1)$ has order at most 2 when q is odd, we have the following special case of Theorem JW. Note, however, that we do not require E_1 to be a fibre bundle.

PROPOSITION A.1. *Let E_2 be an orthogonal q -sphere bundle over S^q where q is odd. If $E_1 \simeq E_2$ then E_1, E_2 are fibre homotopy equivalent.*

Next we consider the case where q is even. We have $(\pi_1\gamma\iota_2)^*\xi = m\xi_2$, and we let $(\pi_1\gamma d_2)^*\xi = n\xi$, where s_2 now denotes a cross-section of E_2 . Let $a = \pi_2^*\xi$ and choose $a_2 \in H^q E_2$ as in I, §2 so that $s_2^*a_2 = 0$ and $\iota_2^*a_2 = \xi_2$. Then $\gamma^*\pi_1^*\xi = na + ma_2$ so $(m, n) = 1$. Since q is even we have $m(2na \cdot a_2 + ma_2^2) = \gamma^*\pi_1^*\xi^2 = 0$, so either

- (i) $m = 0$, or
- (ii) $m = \pm 1$, or
- (iii) $m = \pm 2$.

In case (i) we may again adapt the proof of Theorem 2, to obtain

PROPOSITION A.2. *Let $B = S^q$ where q is even. If $E_1 \simeq E_2$ then there is a homotopy equivalence $f: S^q \rightarrow S^q$ such that E_1, f^*E_2 have the same fibre homotopy type.*

In this way we recover the remainder of Theorem JW. However we do not require E_1, E_2 to be fibre bundles. We conclude by proving Proposition A.2 in cases (ii) and (iii).

In case (ii) both E_1, E_2 are fibre homotopy trivial. In case (iii) there is a fibre preserving map

$$\begin{array}{ccc} E_2 & \xrightarrow{\pi_2 \times \pi_1 \gamma} & S^q \times S^q \\ & \searrow \pi_2 & \swarrow \text{pr}_1 \\ & S^q & \end{array}$$

which is of degree ± 2 on fibres. Let $\alpha \in \pi_{q-1}G_1^q$ classify E_2 . Then a calculation using [18 and 19] shows that α has order at most 2. The proof of Proposition A.1 now applies.

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