DISCRETE ORDERED SETS WHOSE COVERING GRAPHS ARE MEDIAN

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ABSTRACT. As is well known the covering graph (= Hasse diagram) of any median semilattice is a median graph, and every median graph is the covering graph of some median semilattice. The purpose of this note is to prove that an ordered set is a median semilattice whenever (i) no interval contains an infinite chain, (ii) each pair of elements is bounded below, and (iii) the covering graph is median.

The covering graphs (alias undirected Hasse diagrams) of finite distributive lattices have been characterized in several ways; see [1, 2, 4]. For instance, S. P. Avann [2] proved the following result: a finite graph G is the covering graph of a finite distributive lattice if and only if G is a median graph with two distinguished vertices 0 and 1 such that every vertex in G lies on a shortest path joining 0 and 1. Recall that a *median graph* is a (not necessarily finite) connected graph in which for any three vertices u, v, w there is exactly one vertex x (the *median* of u, v, w) such that $d(u, v) = d(u, x) + d(x, v)$, $d(v, w) = d(v, x) + d(x, w)$, and $d(u, w) = d(u, x) + d(x, w)$, where d is the distance function of the graph. Every median graph is the covering graph of a median semilattice with least element a , where a is any fixed vertex: for vertices u and v put $u \leq_a v$ if $d(a, v) = d(a, u) + d(u, v)$; see [2, 6]; cf. [5]. Median semilattices are close to distributive lattices and are defined as follows. A meet-semilattice is called a *median semilattice* if every principal ideal is a distributive lattice and any three elements have an upper bound whenever each pair of them is bounded above. The covering graph of a median semilattice S is a median graph provided that S is discrete, i.e. all intervals are finite; see [2]. In general, an ordered set P is called *discrete* if all bounded chains in P are finite. Then from [2] one can derive the following result: a discrete lattice L with 0 is distributive if and only if the covering graph of L is median. In this theorem the requirement that L have a least element can be dropped; see [3]. Here we prove a more general result.

THEOREM. *Let P be a discrete ordered set in which any two elements are bounded below. Then P is a median semilattice if and only if the covering graph of P is median.*

COROLLARY. *A discrete ordered set P is a distributive lattice if and only if (i) any two elements are bounded below and bounded above, and (ii) the covering graph of P is median.*

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FIGURE 1

FIGURE 2

The conditions of the Theorem cannot be relaxed as is shown by Figure 1 [3] and Figure 2.

Henceforth the covering graph of a discrete ordered set P is denoted by $C(P)$. The *segment* of two vertices u and v in a connected graph is the set $u \circ v$ of all vertices x which lie on a shortest path joining u and v ; that is, $d(u, x) + d(x, v) = d(u, v)$.

LEMMA. *Let P be a discrete ordered set such that $C(P)$ is a median graph. Then for any $a < b$ in P , the interval $[a, b] = \{\hat{x} \in P \mid a \leq x \leq b\}$ coincides with the segment $a \circ b$ of $C(P)$.*

PROOF. (i) To prove that $[a, b] \subseteq a \circ b$ for $a < b$, one has to check that any maximal chain between $a < b$ in the ordered set P is a shortest path joining a and b in the graph $C(P)$. This is, in fact, not difficult to show by using induction on the length of the chain; see the first part of the proof of Theorem 4.5 in [3].

(ii) Suppose that $a < b$ in P and $x \circ y \subseteq [x, y]$ for all $x < y$ in P with $d(x, y) < d(a, b)$. Pick any element v of $[a, b]$ which is covered by b . Let $a, z_1, z_2, \dots, z_{n-1}, w, b$ be any shortest path joining a and b . Since $v \in a \circ b$ by (i), the median u of a, v, w belongs to $a \circ v \cap a \circ w \subseteq [a, v] \cap [a, w]$. Then either $u = v = w$ or $d(b, u) = 2$, whence $w \in u \circ b \subseteq [u, b]$ and thus $a \leq w < b$. By assumption, the subpath $a, z_1, \dots, z_{n-1}, w$ is contained in $[a, w]$, and therefore the whole path is contained in $[a, b]$. This proves the required inclusion $a \circ b \subseteq [a, b]$.

PROPOSITION. *Let P be a discrete ordered set such that $C(P)$ is a median graph. Then every interval of P is a finite distributive lattice.*

PROOF. In view of the Lemma every interval $[a, b]$ of P is a segment of the graph $C(P)$. Therefore the covering graph of $[a, b]$ is a median graph. Hence by Avann's theorem (or [3, Theorem 4.5]) $[a, b]$ is a (finite) distributive lattice provided that $[a, b]$ is a lattice. Suppose by way of contradiction that there exists an interval $[a, b]$ of P which is not a lattice. Then choose $v, w, x, y \in [a, b]$ such that v and w are distinct minimal upper bounds of x and y . By the Lemma the median u of a, v, w is a lower bound of v and w . Further, we get that $x, y \in a \circ v \cap a \circ w = a \circ u$, whence u is an upper bound of x and y . This contradicts the minimality of v and w , and we are done.

FIGURE 3

PROOF OF THE THEOREM. Since any two elements of P are bounded below, we infer from the Proposition that P is a meet-semilattice in which every principal ideal is a distributive lattice. Assume that P is not a median semilattice. Then choose $u, v, w \in P$ such that (1) u, v, w have no common upper bound, (2) $u \vee v, u \vee w, v \vee w$ exist, and (3) for $a = u \wedge v \wedge w$, the ternary distance $n = d(a, u) + d(a, v) + d(a, w)$ is minimal with respect to (1) and (2). If, say, u does not cover a , then pick any t with $a < t < u$. Clearly, t is the meet of $u, t \vee v$, and $t \vee w$. Since $t \vee v \vee w$ exists by (3), the elements $u, t \vee v, t \vee w$ are pairwise bounded above. Now, $d(t, u) + d(t, t \vee v) + d(t, t \vee w) < n$, which is in conflict with (3). Therefore $n = 3$, that is, u, v, w cover a . Then the median x of the vertices $u \vee v, u \vee w, v \vee w$ in $C(P)$ must be an upper bound of u, v, w in the semilattice P (see Figure 3), contrary to the assumption. Hence P is a median semilattice.

REFERENCES

1. L. Alvarez, *Undirected graphs realizable as graphs of modular lattices*, Canad. J. Math. **17** (1965), 923–932.
2. S. P. Avann, *Metric ternary distributive semi-lattices*, Proc. Amer. Math. Soc. **12** (1961), 407–414.
3. H.-J. Bandelt and J. Hedlíková, *Median algebras*, Discrete Math. **45** (1983), 1–30.
4. D. Duffus and I. Rival, *Graphs orientable as distributive lattices*, Proc. Amer. Math. Soc. **88** (1983), 197–200.
5. H. M. Mulder and A. Schrijver, *Median graphs and Helly hypergraphs*, Discrete Math. **25** (1979), 41–50.
6. M. Sholander, *Medians, lattices, and trees*, Proc. Amer. Math. Soc. **5** (1954), 808–812.

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