A FINITE GLOBAL AZUMAYA THEOREM IN ADDITIVE CATEGORIES

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ABSTRACT. Let $\mathbf{C}$ be an additive category such that idempotent endomorphisms have kernels, C a class of objects of $\mathbf{C}$ having Dedekind domains as endomorphism rings, and assume that if X and Y are quasi-isomorphic objects of C then $\text{Hom}(X, Y)$ is a torsion-free module over the endomorphism ring of X . If $A \oplus B = C_1 \oplus \cdots \oplus C_n$ with each C_i in C , then $A = A_1 \oplus \cdots \oplus A_m$, where each A_j is locally in C , and $\text{End}(A_j) \simeq \text{End}(C_i)$ for some i . The proof includes a characterization of tiled orders. Moreover, there is a "local" uniqueness for finite direct sums of objects of C .

Let $\mathbf{C}$ be an additive category such that idempotent endomorphisms of objects of $\mathbf{C}$ have kernels and let C be a class of objects of $\mathbf{C}$ with Dedekind domains as endomorphism rings. Two objects A and B of C are *quasi-isomorphic* if there are maps $f \in \text{Hom}(A, B)$ and $g \in \text{Hom}(B, A)$ with $fg \neq 0$. The class C has the *torsion-free-hom condition* if whenever A and B are quasi-isomorphic objects of C then $\text{Hom}(A, B)$ is a torsion-free $E(A)$ -module, where $E(A)$ is the endomorphism ring of A . If A and X are objects of $\mathbf{C}$ such that $E(A)$ and $E(X)$ are integral domains and if P is a prime ideal of $E(A)$ then A is *isomorphic to X at P* if there are maps $f \in \text{Hom}(A, X)$ and $g \in \text{Hom}(X, A)$ with $E(A) = P + E(A)gf$. An object A such that $E(A)$ is a Dedekind domain is *locally in C* if for each nonzero prime ideal P of $E(A)$ there is an object X in C such that A is isomorphic to X at P . Finally, if C is a class of objects of $\mathbf{C}$ such that $E(X)$ is a Dedekind domain for each X in C and if C has the torsion-free-hom condition then C is said to be a *Dedekind class*.

THEOREM I. *Let $\mathbf{C}$ be an additive category such that idempotent endomorphisms have kernels, and let C be a Dedekind class of objects of $\mathbf{C}$. If $A \oplus B = C_1 \oplus \cdots \oplus C_n$ with each C_i in C , then $A = A_1 \oplus \cdots \oplus A_m$, where for each j there is some i with $E(A_j) \simeq E(C_i)$. In particular, each A_j is locally in $\{C_1, \dots, C_n\}$.*

Theorem I is an example of a finite Azumaya theorem, a deduction of properties of $C_1 \oplus \cdots \oplus C_n$ from conditions on the endomorphism rings of the C_i 's [AHR1, WW1]. In this case, each direct sum decomposition has a refinement into a direct sum of indecomposables. Even though direct sum decompositions into indecomposables need not be unique, there is a "local" uniqueness.

COROLLARY II. *Let $\mathbf{C}$ be an additive category such that idempotent endomorphisms have kernels, and let C be a Dedekind class of objects of $\mathbf{C}$. If $\{C_1, \dots, C_m\}$ and $\{D_1, \dots, D_n\}$ are families of objects of C then $C_1 \oplus \cdots \oplus C_m \simeq D_1 \oplus \cdots \oplus D_n$*

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if and only if $m = n$ and for each X in $\{C_1, C_2, \dots, C_m\}$ and each nonzero prime ideal P of $E(X)$, there is a bijection from $\{i \mid X \text{ is isomorphic to } C_i \text{ at } P\}$ to $\{j \mid X \text{ is isomorphic to } D_j \text{ at } P\}$.

Special cases of Theorem I and Corollary II are proved in [AHR1, Theorems A and D], wherein C is assumed to be a class of objects of $\mathbf{C}$ such that $E(X)$ is a principal ideal domain for each X in C . Results of [AHR1] are used to reduce to the *homogeneous case* (the C_i 's are pairwise quasi-isomorphic) which, in turn, is a consequence of a ring theoretic theorem. This approach is in contrast to [AHR1] which uses additive category techniques for the homogeneous case.

The following notation is assumed: D is a division algebra; K is the center of D ; R is a Dedekind domain with quotient field K ; and T is an R -order in $\text{Mat}_m(D)$ (T is an R -subalgebra of $\text{Mat}_m(D)$, T is finitely generated as an R -module and $KT = \text{Mat}_m(D)$, the ring of $m \times m$ matrices with entries in D).

The order T is said to be a *tilted order* if there are nonzero orthogonal idempotents $e_1, \dots, e_m$ of T with $T = e_1T \oplus \dots \oplus e_mT$ (e.g. see [JA1, JA2]). Note that T is a tilted order if and only if there is a K -algebra automorphism ϕ of $\text{Mat}_m(D)$ with $e_{ii} \in \phi(T)$ for each $1 \leq i \leq m$, where e_{ii} is the matrix with 1 in the i, i position and 0 elsewhere. Tiled orders are called graduated orders in [ZA1 and PL1].

For each nonzero prime ideal P of R let R_P denote the localization of R at P . Then $T_P = R_P \otimes_R T$ is an R_P -order in $\text{Mat}_m(D)$.

The ring T is *semiperfect* if $T/J(T)$ is artinian and each idempotent of $T/J(T)$ lifts to an idempotent in T , where $J(T)$ is the Jacobson radical of T .

The following theorem is a global version of a characterization of tiled orders over local Dedekind domains (see [JA2, PL1]).

THEOREM III. *Assume that R is a Dedekind domain, D is a division algebra, T is an R -order in $\text{Mat}_m(D)$, and that T_P is semiperfect for each nonzero prime ideal P of R . Then the following statements are equivalent.*

- (a) *If P is a nonzero prime ideal of R and if f is a primitive idempotent of T_P , then fT_Pf is isomorphic to an R_P -order in D .*
- (b) *If e is a primitive idempotent of T then eTe is isomorphic to an R -order in D .*
- (c) *T is a tiled order.*
- (d) *For each nonzero prime ideal P of R ,*

$$T_P/J(T_P) \simeq \text{Mat}_{n_1}(D_1) \times \dots \times \text{Mat}_{n_k}(D_k),$$

where each D_i is a division algebra and $n_1 + n_2 + \dots + n_k \geq m$.

Note that if T is an R -order in $\text{Mat}_m(K)$ then T_P is semiperfect for each prime ideal P of R (e.g. [ROHD1, IV 12, p. 189]).

Theorem III includes the case that T is a hereditary R -order in $\text{Mat}_m(K)$. On the other hand, there is a Z_p -order in $\text{Mat}_2(Q)$ satisfying Theorem III that is not hereditary (Example 1.2). Not every Z -order in $\text{Mat}_m(Q)$ satisfies Theorem III (Example 1.3).

1. Orders in matrix rings over division algebras.

PROOF OF THEOREM III. (a) $\Rightarrow$ (b) Let e be a primitive idempotent of T and let P be a nonzero prime ideal of R . There is a set $\{e_1, \dots, e_k\}$ of primitive orthogonal idempotents of T_P with $T_P = e_1T_P \oplus \dots \oplus e_kT_P$, $e_iT_Pe_i$ isomorphic to an R_P -order in D for each i , and $e = e_1 + e_2 + \dots + e_k$. Then each $e_iKT_Pe_i \simeq D$, so that $\text{Mat}_m(D) =$

$KT_P = e_1KT_P \oplus \cdots \oplus e_kKT_P$ and $k = m$. Thus, there is a K -algebra automorphism ϕ_P of $\text{Mat}_m(D)$ with $e_{ii} \in \phi_P(T_P)$ for each $1 \leq i \leq m$ and $\phi_P(e) = e_{11} + \cdots + e_{ll}$. By the Skolem-Noether theorem (e.g. [RE1]) there is a unit u_P of $\text{Mat}_m(D)$ with $\phi_P(x) = u_P x u_P^{-1}$ for each $x \in T_P$.

Let $f_P = u_P^{-1}e_{11}u_P = \phi_P^{-1}(e_{11}) \in T_P$. Then f_P is an idempotent of T_P and $ef_Pe = f_P$, since

$$\begin{aligned} \phi_P(ef_Pe) &= \phi_P(e)\phi_P(f_P)\phi_P(e) = (e_{11} + \cdots + e_{ll})e_{11}(e_{11} + \cdots + e_{ll}) \\ &= e_{11} = \phi_P(f_P). \end{aligned}$$

Thus, $f_P T_P$ is isomorphic to a T_P -summand of eT_P . But

$$e_{11}T_P = u_P f_P u_P^{-1} T_P \simeq f_P u_P^{-1} T_P \simeq f_P T_P$$

as right T_P -modules. Consequently, $e_{11}T_P$ is isomorphic to a T_P -module summand of eT_P for each nonzero prime ideal P of R . Hence, $eT = M \oplus N$ for some right T -modules M and N with $M_P \simeq e_{11}T_P$ for each nonzero prime ideal P of R [SE1, Corollary 6.13]. Since e is primitive, $eT = M$ and $e_{11}T_P \simeq eT_P$ for each nonzero prime ideal P . Hence, $eTe \subseteq eT_P e \simeq e_{11}T_P e_{11} \subseteq D$, so that eTe is isomorphic to an R -order in D .

(b) $\Rightarrow$ (c) As noted earlier, $T = e_1T \oplus \cdots \oplus e_kT$, with each e_iTe_i isomorphic to an R -order in D , implies that $k = m$.

(c) $\Rightarrow$ (d) Since T is a tiled order, T_P is a tiled order, say $T_P = e_1T_P \oplus \cdots \oplus e_mT_P$. Then

$$T_P/J(T_P) = (e_1T_P + J(T_P))/J(T_P) \oplus \cdots \oplus (e_mT_P + J(T_P))/J(T_P)$$

is a semisimple artinian ring with each $e_i + J(T_P) \neq 0$. Hence $T_P/J(T_P)$ has at least m simple summands so that $T_P/J(T_P) = \text{Mat}_{n_1}(D_1) \times \cdots \times \text{Mat}_{n_k}(D_k)$ with $n_1 + \cdots + n_k \geq m$ and each D_i a division algebra.

(d) $\Rightarrow$ (a) Let P be a nonzero prime ideal of R . Since T_P is semiperfect, there is a set of orthogonal idempotents $e_1, \dots, e_m$ with $T_P = e_1T_P \oplus \cdots \oplus e_mT_P$ (lift m of the diagonal idempotents from $T_P/J(T_P)$, e.g. [AF1, Proposition 27.4]). Then $e_iT_P e_i$ is isomorphic to an R_P -order in D so that each e_i is primitive. Thus $e_iT_P e_i \simeq \text{End}_{T_P}(e_iT_P)$ is a local ring [AF1, Theorem 27.6]. Now let f be a primitive idempotent of T_P . Then $fT_P \simeq e_iT_P$ for some i (the classical local Azumaya Theorem, e.g. [AF1, Theorem 12.6]). Thus, $fT_P f \simeq e_iT_P e_i$ is isomorphic to an R_P -order in D as desired.

COROLLARY 1.1. *Assume that R is a Dedekind domain with quotient field K and that T is an R -order in $\text{Mat}_m(K)$. If T is a tiled order and e is a primitive idempotent of T then $eTe \simeq R$.*

PROOF. By Theorem III, and the remark following Theorem III, eTe is isomorphic to an R -order S in K . Since R is Dedekind and $R \subseteq S \subseteq K$, the quotient field of R , it follows that $R \simeq eTe$.

EXAMPLE 1.2. Given a prime p of Z ,

$$T = \begin{pmatrix} Z & Z \\ p^2Z & Z \end{pmatrix}$$

is a Z -order in $\text{Mat}_2(Q)$ such that if e is a primitive idempotent of T then $eTe \simeq Z$ but T is not a hereditary Z -order in $\text{Mat}_2(Q)$.

EXAMPLE 1.3. Let $T = \{ \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \mid a \equiv d \pmod{p}, a, b, c, d \in Z_p \}$. Then $T = T_p$ is semiperfect, 1 is a primitive idempotent in T , and T is not isomorphic to Z_p .

2. A finite global Azumaya theorem.

LEMMA 2.1. *Let $G = C_1 \oplus \cdots \oplus C_m$, where the C_i 's are pairwise quasi-isomorphic objects of a Dedekind class C . Then there is a Dedekind domain R with quotient field K and an R -order T in $\text{Mat}_m(K)$ such that each $E(C_i)$ may be identified with R , $E(G) \simeq T$, and T is a tiled order.*

PROOF. A consequence of [AHR1, Theorems 2.1, 2.2 and Corollary 2.3].

LEMMA 2.2. *Let $\mathbf{C}$ be an additive category such that idempotent endomorphisms of objects of $\mathbf{C}$ have kernels, and let C be a Dedekind class of objects of $\mathbf{C}$. The following are equivalent.*

- (a) *If $G = A \oplus B = C_1 \oplus \cdots \oplus C_n$ with each C_i in C then $A = A_1 \oplus \cdots \oplus A_m$, where for each j there is some i with $E(A_j) \simeq E(C_i)$.*
- (b) *If $G = A \oplus B = C_1 \oplus \cdots \oplus C_n$ with each C_i in C and the C_i 's are pairwise quasi-isomorphic then $A = A_1 \oplus \cdots \oplus A_m$, where for each j there is some i with $E(A_j) \simeq E(C_i)$.*
- (c) *If $G = C_1 \oplus \cdots \oplus C_n$ with C_i in C , the C_i 's are pairwise quasi-isomorphic, and e is a primitive idempotent in $E(G)$, then $eE(G)e \simeq E(C_i)$ for some i .*

PROOF. (a) $\Rightarrow$ (b) is clear.

(b) $\Rightarrow$ (a) Group the C_i 's by quasi-isomorphism classes to get $G = A \oplus B = K_1 \oplus \cdots \oplus K_l$. Then $A = A_1 \oplus \cdots \oplus A_l$, where each A_i is a summand of K_i [AHR1, Corollary 2.6]. Now apply (b).

(b) $\Rightarrow$ (c) Note that $G = e(G) \oplus (1 - e)(G)$, where $e(G) = \text{Kernel}(1 - e)$ and $(1 - e)(G) = \text{Kernel}(e)$. Since e is primitive, $e(G)$ is indecomposable so that $eE(G)e \simeq E(e(G)) \simeq E(C_i)$ for some i .

(c) $\Rightarrow$ (b) If A_1 is an indecomposable summand of A then $A_1 = e(G)$ for some primitive idempotent e of $E(G)$. Thus $E(A_1) \simeq E(e(G)) \simeq eE(G)e \simeq E(C_i)$ for some i . Finally, by Lemma 2.1, there is a Dedekind domain R with $R \simeq E(C_i)$ for each i and $E(G)$ is a finite rank torsion-free R -module. Consequently, $A = A_1 \oplus \cdots \oplus A_m$ with each A_j indecomposable. Therefore, $E(A_j) \simeq R \simeq E(C_i)$ for each j .

PROOF OF THEOREM I. Assume that $A \oplus B = C_1 \oplus \cdots \oplus C_n$ with each C_i in C . By Lemma 2.2, it is sufficient to assume that the C_i 's are pairwise quasi-isomorphic. Now apply Lemma 2.1, Corollary 1.1, and Lemma 2.2 to see that $A = A_1 \oplus \cdots \oplus A_n$, where for each j there is some i with $E(A_j) \simeq E(C_i)$, a Dedekind domain. Let $X = A_j$. There are maps $f_k \in \text{Hom}(X, C_k)$ and $g_k \in \text{Hom}(C_k, X)$ with $1_X = g_1 f_1 + \cdots + g_n f_n$, since X is a summand of $C_1 \oplus \cdots \oplus C_n$. If P is a nonzero prime ideal of $E(X)$ then some $g_k f_k \notin P$. Thus, $E(X) = P + E(X)g_k f_k$ and $X = A_j$ is locally in $\{C_1, C_2, \dots, C_n\}$.

PROOF OF COROLLARY II. ($\Rightarrow$) Assume that

$$G = C_1 \oplus \cdots \oplus C_m \simeq D_1 \oplus \cdots \oplus D_n$$

with each C_i and D_j in C . It suffices to assume that the C_i 's and D_j 's are all quasi-isomorphic by grouping together the C_i 's and D_j 's in quasi-isomorphism classes and applying [AHR1, Corollary 2.6 and Theorem 2.7]. By Lemma 2.2, there is a Dedekind domain R with quotient field K and an R -order T in $\text{Mat}_m(K)$ such that $E(C_i) = R$ for each i , $E(G) \simeq T$, and T is a tiled order.

If P is a nonzero prime ideal of R then R_P is a local ring and T_P is an R_P -order in $\text{Mat}_m(K)$ with a full set of idempotents. Choose idempotents e_i and f_j of T such that $1 = e_1 + \cdots + e_m = f_1 + \cdots + f_n$, $C_i \simeq e_i G = \text{Kernel}(1 - e_i)$, and $D_j \simeq f_j G = \text{Kernel}(1 - f_j)$. Then $E(D_j) \simeq f_j T f_j \simeq R$, by Theorem I, since $E(D_j)$ a domain implies that f_j is primitive. Hence, $m = n$.

Now $T_P = e_1 T_P \oplus \cdots \oplus e_m T_P = f_1 T_P \oplus \cdots \oplus f_m T_P$ and $E(e_i T_P) \simeq R_P E(f_j T_P)$ is a local ring. By the classical local Azumaya theorem, there is a permutation σ of $\{1, 2, \dots, m\}$ such that $e_i T_P \simeq f_{\sigma(i)} T_P$ for each i . Thus, there is

$$h'_i: e_i T \rightarrow f_{\sigma(i)} T \quad \text{and} \quad g'_i: f_{\sigma(i)} T \rightarrow e_i T$$

with $E(e_i T) = P + E(e_i T)g'_i h'_i$, noting that $E(e_i T) = e_i T e_i = E(C_i) = R$. Then

$$h_i = f_{\sigma(i)} h'_i e_i: C_i \rightarrow D_{\sigma(i)} \quad \text{and} \quad g_i = e_i g'_i f_{\sigma(i)}: D_{\sigma(i)} \rightarrow C_i$$

with $E(C_i) = P + E(C_i)g_i h_i$. Consequently, σ induces a bijection from $\{i \mid X \text{ is isomorphic to } C_i \text{ at } P\}$ to $\{j \mid X \text{ is isomorphic to } D_j \text{ at } P\}$.

($\Leftarrow$) It suffices to assume that the C'_i 's and D'_j 's are all quasi-isomorphic. By Lemma 2.2, there is a Dedekind domain R with $E(D_j) = R$ for each j . Since C_1 is locally in $\{D_1, D_2, \dots, D_n\}$,

$$D_1 \oplus \cdots \oplus D_n = C_1 \oplus B_2 \oplus \cdots \oplus B_k$$

for some B_i (as in [AHR1, Lemma 3.1]). By Theorem I, it suffices to assume that $E(B_i) = R$ for $2 \leq i \leq k$ and that each B_i is locally in $\{D_1, D_2, \dots, D_n\}$. As a consequence of ($\Rightarrow$) and the hypothesis, for each X in $\{C_1, C_2, \dots, C_m\}$ and each prime ideal P of $E(X)$ there is a bijection from $\{i \neq 1 \mid X \text{ is isomorphic to } C_i \text{ at } P\}$ to $\{j \mid X \text{ is isomorphic to } B_j \text{ at } P\}$. By induction on the number of C'_i 's, $C_2 \oplus \cdots \oplus C_m \simeq B_2 \oplus \cdots \oplus B_k$ and so $C_1 \oplus \cdots \oplus C_m \simeq D_1 \oplus \cdots \oplus D_n$, as desired.

REMARK. Applications of Theorem I and Corollary II include the applications listed in [AHR1, §6], with appropriate and obvious modifications, the finite rank case of [KO1, Theorem 8], and a generalization of [AR1, Theorem 6.11].

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