EXTREME POINTS OF SUBORDINATION FAMILIES WITH UNIVALENT MAJORANTS

DAVID J. HALLENBECK

ABSTRACT. Let $s(F)$ denote the set of functions subordinate to a univalent function F in Δ the unit disc. Let B_0 denote the set of functions $\phi(z)$ analytic in Δ satisfying $|\phi(z)| < 1$ and $\phi(0) = 0$. We prove the following results: If $f = F \circ \phi$ is an extreme point of $s(F)$ and $F(\Delta)$ is a Jordan domain, then ϕ is an extreme point of B_0 .

1. Introduction. Let $\Delta = \{z: |z| < 1\}$ and let $\mathcal{A}$ denote the set of functions analytic in Δ . Let B_0 consist of the subset of $\mathcal{A}$ consisting of all functions ϕ that satisfy the conditions $|\phi(z)| < 1$, $\phi(0) = 0$. Let EB_0 denote the extreme points of B_0 . It is a classical fact [2, p. 125] that $\phi \in EB_0$ if and only if $\int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|) d\theta = -\infty$. Let S denote the subset of $\mathcal{A}$ consisting of univalent functions f so that $f(z) = z + \dots$ in Δ .

Let F be in $\mathcal{A}$ and be univalent in Δ . Let $s(F)$ denote the subset of $\mathcal{A}$ consisting of functions f that are subordinate to F in Δ . This means that $f \in \mathcal{A}$, $f(0) = F(0)$, and $f(\Delta) \subset F(\Delta)$. These conditions are equivalent to the existence of $\phi \in B_0$ so that $f = F \circ \phi$. Note that $s(F) = \{F \circ \phi: \phi \in B_0\}$.

Let D denote $F(\Delta)$. Since $F \in H^p$ for $p < \frac{1}{2}$ [2, p. 50], if we define $f(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta})$ then $f(e^{i\theta})$ exists almost everywhere. Let $\lambda(\theta)$ denote the distance between $f(e^{i\theta})$ and ∂D where ∂D denotes the boundary of D .

We let $Hs(F)$ and $EHs(F)$ denote respectively the closed convex hull of $s(F)$ and the extreme points of the closed convex hull of $s(F)$. Since $s(F)$ is compact, $EHs(F) \subset s(F)$. Let $Es(F)$ denote the extreme points of $s(F)$. In [1], Yusuf Abu-Muhanna proved if F' is Nevanlinna and D a Jordan domain subset to a half plane then $EHs(F) \subset \{F \circ \phi: \phi \in EB_0\}$. He conjectured that the inclusion holds for any univalent function F . We prove this inclusion only assuming F is univalent and D is a Jordan domain. Our arguments are similar to those of Abu-Muhanna [1].

2. Functions subordinate to a univalent function. We let $d(z, \Gamma)$ denote the distance between z and a closed set Γ , $m(A)$ denote the Lebesgue measure of A and $\log^+ x = \max\{0, \log x\}$.

THEOREM 1. *Let F be in $\mathcal{A}$ and be univalent in Δ . Whenever g is in $s(F)$, let $\lambda(\theta)$ denote the distance between $g(e^{i\theta})$ and ∂D where $D = F(\Delta)$. Then when $g = F \circ \phi$,*

(a) $\int_0^{2\pi} \log^+ \lambda(\theta) d\theta$ *is convergent and*

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(b) $\int_0^{2\pi} \log \lambda(\theta) d\theta = -\infty$ implies $\int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|) d\theta = -\infty$.
 If F' is Nevanlinna then

(c) $\int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|) d\theta = -\infty$ implies $\int_0^{2\pi} \log \lambda(\theta) d\theta = -\infty$.

PROOF. Since F is univalent, it follows from [3, p. 22] that

$$(1) \quad \frac{1}{4}(1 - |z|^2)|F'(z)| \leq d(F(z), \partial D) \leq (1 - |z|^2)|F'(z)|$$

for all z in Δ . For g in $s(F)$ we have $g = F \circ \phi$ where $\phi \in B_0$. If $|\phi(e^{i\theta})| = 1$ for some θ the univalence of F gives $\lambda(\theta) = 0$ and $\log^+ \lambda(\theta) = 0$. When $g(e^{i\theta})$ and $\phi(e^{i\theta})$ exist, and $|\phi(e^{i\theta})| < 1$, we obtain from (1)

$$(2) \quad \frac{1}{4}(1 - |\phi(e^{i\theta})|^2)|F'(\phi(e^{i\theta}))| \leq \lambda(\theta) \leq (1 - |\phi(e^{i\theta})|^2)|F'(\phi(e^{i\theta}))|,$$

which implies $\lambda(\theta) \leq 2(1 - |\phi(e^{i\theta})|)|F'(\phi(e^{i\theta}))|$. We may assume, without loss of generality, that $F \in S$ and so, it follows that

$$|F'(z)| \leq \left| \frac{F(z)}{z} \right| \frac{1 + |z|}{1 - |z|}$$

for z in Δ [3, p. 21]. Hence,

$$2(1 - |\phi(e^{i\theta})|)|F'(\phi(e^{i\theta}))| \leq 4 \left| \frac{F(\phi(e^{i\theta}))}{\phi(e^{i\theta})} \right|$$

and so

$$0 \leq \log^+ \lambda(\theta) \leq \log^+ 4 \left| \frac{F(\phi(e^{i\theta}))}{\phi(e^{i\theta})} \right|.$$

It is known that $F(z)/z$ is in H^p for each $p < \frac{1}{2}$ and so we have $F(\phi(z))/\phi(z)$ in H^p for each $p < \frac{1}{2}$. It follows that $\log^+ |F(\phi(e^{i\theta}))/\phi(e^{i\theta})|$ is integrable on $[0, 2\pi]$. It follows directly that (a) holds.

Now let $A = \{\theta: g(e^{i\theta}) \text{ exists and } \lambda(\theta) = 0\}$. If $m(A) > 0$ then it follows that $\int_0^{2\pi} \log \lambda(\theta) d\theta = -\infty$. It also follows that $\int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|) d\theta = -\infty$ since $g(e^{i\theta})$ on ∂D implies that $|\phi(e^{i\theta})| = 1$, given that $F \in S$. Assume for the rest of the proof that $m(A) = 0$. This implies (2) holds for almost all θ . The fact that $1 \leq 1 + |\phi(e^{i\theta})| \leq 2$ applied to (2) gives

$$(3) \quad \frac{1}{4}(1 - |\phi(e^{i\theta})|)|F'(\phi(e^{i\theta}))| \leq \lambda(\theta) \leq 2(1 - |\phi(e^{i\theta})|)|F'(\phi(e^{i\theta}))|$$

for almost all θ . Since F is in S , we have [3, p. 21] $|F'(z)| \geq (1 - |z|)/(1 + |z|)^3$ for z in Δ . Hence

$$|F'(\phi(e^{i\theta}))| \geq \frac{1}{8}(1 - |\phi(e^{i\theta})|)$$

and this fact, taken together with the left-hand side of (3) gives

$$\lambda(\theta) \geq \frac{1}{4}(1 - |\phi(e^{i\theta})|)|F'(\phi(e^{i\theta}))| \geq \frac{1}{32}(1 - |\phi(e^{i\theta})|)^2.$$

This implies

$$(4) \quad -\log 32 + 2 \log(1 - |\phi(e^{i\theta})|) \leq \log \lambda(\theta).$$

Since $\log^+ \lambda(\theta) \in L^1[0, 2\pi]$ we have

$$(5) \quad -\infty \leq \int_0^{2\pi} \log \lambda(\theta) d\theta < M$$

for some constant M . If $\int_0^{2\pi} \log \lambda(\theta) d\theta = -\infty$ then (4) and (5) imply

$$\int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|) d\theta = -\infty$$

and (b) holds.

The proof of (c) is given in [1, p. 440].

3. Jordan domains. The following lemma appeared in [1, p. 444].

LEMMA A. *Let D be a bounded Jordan domain. Let g be a nonconstant bounded analytic function in Δ . If $g(e^{i\theta}) \in \bar{D}$ for almost all θ then $g(\theta) \subset D$.*

LEMMA 1. *Let D be an unbounded Jordan domain. Let g be a nonconstant analytic function in Δ so that $g(\Delta) \cup D$ avoids an open set. If $g(e^{i\theta}) \in \bar{D}$ for almost all θ then $g(\Delta) \subset D$.*

PROOF. There exists a complex number a and a $\delta > 0$ so that $\{w: |w - a| < \delta\} \cap \{g(\Delta) \cup D\} = \emptyset$. It follows that the function $h(z) = 1/(g(z) - a)$ is analytic and bounded for $z \in \Delta$. Also, $h(e^{i\theta}) = 1/(g(e^{i\theta}) - a)$ exists for almost all θ and $h(e^{i\theta}) \in \overline{f(D)}$ where $f(w) = 1/(w - a)$. It follows from Lemma A that $h(\Delta) \subset f(D)$ or $f(g(\Delta)) \subset f(D)$. Hence $g(\Delta) \subset D$.

THEOREM 2. *Let F be in $\mathcal{Q}$ and univalent in Δ and suppose that $D = F(\Delta)$ is a Jordan domain. Let $\lambda(\theta)$ denote the distance between $f(e^{i\theta})$ and ∂D where $f \in s(F)$. If $f \in Es(F)$, then*

$$(6) \quad \int_0^{2\pi} \log \frac{\lambda(\theta)}{1 + \lambda(\theta)} d\theta = -\infty.$$

Also, if F is bounded then (6) is equivalent to

$$(7) \quad \int_0^{2\pi} \log \lambda(\theta) d\theta = -\infty.$$

PROOF. We assume

$$\int_0^{2\pi} \log \frac{\lambda(\theta)}{1 + \lambda(\theta)} d\theta > -\infty$$

and show that $f \notin Es(F)$. Since $\lambda(\theta)/(1 + \lambda(\theta)) < 1$ our assumption implies by $\log(\lambda(\theta)/(1 + \lambda(\theta)))$ is integrable on $[0, 2\pi]$. Let $d(z; D)$ denote the distance between z and D . Since D is a Jordan domain, there exists ϵ , $0 < \epsilon < 1$, such that $E = \{z: d(z, D) \leq \epsilon\}$ avoids an open set. The function $\log \frac{\epsilon}{2} \frac{\lambda(\theta)}{1 + \lambda(\theta)}$ is also integrable on $[0, 2\pi]$. Let

$$(8) \quad g(z) = z \exp \left[\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \frac{\epsilon}{2} \frac{\lambda(t)}{1 + \lambda(t)} dt \right].$$

(8) gives $|g(z)| < \varepsilon/2$ for $z \in \Delta$ and it is known that

$$|g(e^{i\theta})| = \frac{\varepsilon}{2} \frac{\lambda(\theta)}{1 + \lambda(\theta)}$$

for almost all θ . Since $\lambda(\theta)$ is the distance between $f(e^{i\theta})$ and ∂D and $|g(e^{i\theta})| < \lambda(\theta)/2$, it follows that $f(e^{i\theta}) \pm g(e^{i\theta}) \in \bar{D}$ for almost all θ . Note that $g \neq 0$. If $h = f \pm g$ then $|g(e^{i\theta})| < \varepsilon/2$ implies $h(\Delta) \subset E$. Lemma 1 now implies $f(z) \pm g(z) \in D$ for all $z \in \Delta$. Since $g(0) = 0$ and $f \in s(F)$, we conclude $f \pm g \in s(F)$ and so $f \notin Es(F)$. This completes the proof.

REMARK. Since $EHs(F) \subset Es(F)$, (6) holds if $f \in EHs(F)$.

THEOREM 3. Let $F \in \mathcal{Q}$ be univalent in Δ . If $D = F(\Delta)$ is a Jordan domain then $Es(F) \subset \{F \circ w: w \in EB_0\}$.

PROOF. Let $g = F \circ w \in Es(F)$ where $w \in B_0$. We shall prove that

$$\int_0^{2\pi} \log(1 - |w(e^{i\theta})|) d\theta = -\infty$$

and so conclude that $w \in EB_0$. If we let $\lambda(\theta)$ be the distance between $g(e^{i\theta})$ and ∂D , then Theorem 2 gives

$$\int_0^{2\pi} \log \frac{\lambda(\theta)}{1 + \lambda(\theta)} d\theta = -\infty.$$

Let $E = \{\theta: \lambda(\theta) \text{ exist and } \lambda(\theta) \leq 1\}$ and let $G = \{\theta: \lambda(\theta) \text{ exist and } \lambda(\theta) > 1\}$. Clearly $m(E \cup G) = 2\pi$. For $\theta \in E$, we have

$$(9) \quad \frac{\lambda(\theta)}{2} \leq \frac{\lambda(\theta)}{1 + \lambda(\theta)} \leq \lambda(\theta),$$

and for $\theta \in G$, we have $1 + \lambda(\theta) < 2\lambda(\theta)$ and so

$$(10) \quad \frac{1}{2} < \frac{\lambda(\theta)}{1 + \lambda(\theta)} < 1.$$

Equation (10) implies

$$\int_G \log \frac{\lambda(\theta)}{1 + \lambda(\theta)} d\theta$$

is convergent. Therefore,

$$\int_0^{2\pi} \log \frac{\lambda(\theta)}{1 + \lambda(\theta)} d\theta = \int_E \log \frac{\lambda(\theta)}{1 + \lambda(\theta)} d\theta = -\infty$$

and by (9) $\int_E \log \lambda(\theta) d\theta = -\infty$. Now

$$(11) \quad \int_0^{2\pi} \log \lambda(\theta) d\theta = \int_G \log \lambda(\theta) d\theta + \int_E \log \lambda(\theta) d\theta.$$

By Theorem 1, $\int_0^{2\pi} \log^+ \lambda(\theta) d\theta$ is convergent and so $\int_G \log \lambda(\theta) d\theta = \int_G \log^+ \lambda(\theta) d\theta$ is convergent. Since $\int_E \log \lambda(\theta) d\theta = -\infty$ we conclude from (11) that $\int_0^{2\pi} \log \lambda(\theta) d\theta = -\infty$. Again, by Theorem 1, we conclude that

$$\int_0^{2\pi} \log(1 - |w(e^{i\theta})|) d\theta = -\infty.$$

Hence $w \in EB_0$.

REMARK. We note that $EHs(F) \subset \{F \circ w: w \in EB_0\}$ since $EHs(F) \subset Es(F)$.

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF DELAWARE, NEWARK, DELAWARE 19711