BLOCH CONSTANTS FOR MEROMORPHIC FUNCTIONS NEAR AN ISOLATED SINGULARITY

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ABSTRACT. Suppose f is meromorphic in a punctured neighborhood of the origin and has an essential singularity at the origin. Given any $\varepsilon > 0$ we show that the Riemann surface of f contains an unramified disk of spherical radius $\pi/3 - \varepsilon$. The number $\pi/3$ can be replaced by $\pi/2$ if f is locally schlicht and this value is best possible. If f is actually holomorphic, then the Riemann surface of f contains arbitrarily large unramified euclidean disks. These results generalize theorems of Valiron and Ahlfors dealing with holomorphic and meromorphic functions, respectively, on the complex plane which have an essential singularity at infinity.

1. Introduction. In considering the behavior of a meromorphic function in a neighborhood of an isolated singularity, we shall normalize to the case of a function meromorphic on the punctured unit disk $\mathbf{D}^* = \{z: 0 < |z| < 1\}$. We begin by briefly establishing some notation; for more details see [7 or 8]. For f meromorphic on $\mathbf{D}^*$, let R_f denote the Riemann surface of f , viewed as spread over the Riemann sphere $\mathbf{P}$. For $z \in \mathbf{D}^*$ let $r_{\mathbf{P}}(z, f)$ denote the spherical radius of the largest unramified disk in R_f with center $f(z)$. Of course, $r_{\mathbf{P}}(z, f) = 0$ if $f(z)$ is a branch point of R_f . Let $r_{\mathbf{P}}(f) = \sup\{r_{\mathbf{P}}(z, f): z \in \mathbf{D}^*\}$. If f is actually holomorphic in $\mathbf{D}^*$, then we regard R_f as being spread over the complex plane $\mathbf{C}$. In this situation, $r_{\mathbf{C}}(z, f)$ designates the euclidean radius of the largest unramified disk in R_f with center $f(z)$ and $r_{\mathbf{C}}(f) = \sup\{r_{\mathbf{C}}(z, f): z \in \mathbf{D}^*\}$.

Now, assume f has an essential singularity at the origin. We shall derive Bloch constants, that is, positive lower bounds independent of f , for $r_{\mathbf{C}}(f)$ and $r_{\mathbf{P}}(f)$, depending on whether f is holomorphic or meromorphic in $\mathbf{D}^*$. This localizes to the case of an isolated essential singularity known results for transcendental entire and meromorphic functions on $\mathbf{C}$. If f is holomorphic in $\mathbf{D}^*$, we show that $r_{\mathbf{C}}(f) = \infty$. Valiron established the analogous result for transcendental entire functions; Bloch's theorem for normalized holomorphic functions in the unit disk is a generalization of this result [3]. In case f is meromorphic in $\mathbf{D}^*$ we obtain $r_{\mathbf{P}}(f) \geq \pi/3$. This is improved to the sharp inequality $r_{\mathbf{P}}(f) \geq \pi/2$ when f is also locally schlicht. For a meromorphic function on $\mathbf{C}$ with an essential singularity at ∞ , Ahlfors [1] showed that its Riemann surface contains an unramified disk of spherical radius $\pi/4 - \varepsilon$ for any $\varepsilon > 0$. Minda [8] showed that $\pi/4$ could be replaced by $\pi/3$ in the general case and by $\pi/2$ for locally schlicht meromorphic functions on $\mathbf{C}$.

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For the case of meromorphic functions one of the main ingredients in our proof is a result of Lehto and Virtanen [4–6] dealing with the spherical derivative of a meromorphic function near an isolated essential singularity. There is a slight error in their proof; we pinpoint the error and correct it in the final section.

2. Main results. We shall need the hyperbolic metric with constant curvature -1 on $\mathbf{D}^*$; it is [2, p. 17]

$$\lambda_{\mathbf{D}^*}(z)|dz| = \frac{|dz|}{|z| \log(1/|z|)}.$$

THEOREM 1. *Suppose f is holomorphic in $\mathbf{D}^*$ and has an essential singularity at the origin. Then $r_{\mathbf{C}}(f) = \infty$.*

PROOF. Suppose the conclusion of the theorem were false. Set $\tau = r_{\mathbf{C}}(f) < \infty$. Then for any $z \in \mathbf{D}^*$ [7],

$$\frac{2|f'(z)|}{\lambda_{\mathbf{D}^*}(z)} < \frac{4\tau}{\sqrt{3}}, \quad |zf'(z)| < \frac{2\tau}{\sqrt{3} \log(1/|z|)}.$$

This gives $\lim_{z \rightarrow 0} zf'(z) = 0$, which implies f' , and hence f , has a removable singularity at the origin, a contradiction.

For any positive integer m let $\mathfrak{F}_m(\mathbf{D}^*)$ denote the family of all nonconstant meromorphic functions f on $\mathbf{D}^*$ with the property that for each $w \in f(\mathbf{D}^*)$ any root of $f(z) = w$ is either simple or else has multiplicity at least $m + 1$. Observe that $\mathfrak{F}_1(\mathbf{D}^*)$ is just the family of all nonconstant meromorphic functions on $\mathbf{D}^*$. Let $\mathfrak{F}_{\infty}(\mathbf{D})$ be the family of all locally schlicht meromorphic functions on $\mathbf{D}^*$.

THEOREM 2. *Suppose $f \in \mathfrak{F}_m(\mathbf{D}^*)$, $m \in \mathbf{Z}^+ \cup \{\infty\}$, has an essential singularity at the origin. Then $r_{\mathbf{P}}(f) \geq 2 \tan^{-1}(\sqrt{m/(m+2)})$.*

PROOF. We shall show that the assumption that $f \in \mathfrak{F}_m(\mathbf{D}^*)$ satisfies $r_{\mathbf{P}}(f) < 2 \tan^{-1}(\sqrt{m/(m+2)})$ leads to a contradiction. Assume $r_{\mathbf{P}}(f) = 2 \tan^{-1}(\tau)$, where $\tau < \sqrt{m/(m+2)}$. Then for any $z \in \mathbf{D}^*$ [7],

$$2f^*(z)/\lambda_{\mathbf{D}^*}(z) < 1/\psi_m(\tau),$$

where $f^*(z)$ is the spherical derivative of f and

$$\psi_m(\tau) = \begin{cases} \frac{(m+2-m\tau^2)^{1/2}(m-(m+2)\tau^2)^{1/2}}{2(m+1)\tau}, & m \in \mathbf{Z}^+, \\ (1-\tau^2)/2\tau, & m = \infty. \end{cases}$$

As in the proof of Theorem 1, we obtain $\lim_{z \rightarrow 0} |z|f^*(z) = 0$. However, Lehto and Virtanen [4–6] have shown that

$$\limsup_{z \rightarrow 0} |z|f^*(z) \geq 1/2$$

if f has an isolated essential singularity at the origin. (There is a slight gap in their proof; it will be rectified in the next section.) This contradiction establishes the theorem.

For $m = 1$ we obtain $r_{\mathbf{P}}(f) \geq \pi/3 = 60^\circ$, while for $m = \infty$ we get $r_{\mathbf{P}}(f) \geq \pi/2$. The function $f(z) = \exp(1/z)$ belongs to $\mathcal{F}_\infty(\mathbf{D}^*)$ and $r_{\mathbf{P}}(f) = \pi/2$, so Theorem 2 is sharp in case $m = \infty$. By making use of an example in [8] it is possible to construct a function $f \in \mathcal{F}_1(\mathbf{D}^*)$ with $r_{\mathbf{P}}(f) = 2 \arctan(1/\sqrt{2}) = 70^\circ 32'$.

3. The spherical derivative near an isolated singularity. Lehto and Virtanen [6] demonstrated that there is an absolute positive constant k such that

$$\limsup_{z \rightarrow 0} |z|f^\#(z) \geq k$$

for any meromorphic function f which has an isolated essential singularity at the origin. Lehto [4, 5] demonstrated that it was possible to take $k = 1/2$ and that this choice of k was the best possible. Lehto's "proof" that $k = 1/2$ is short and elegant, so it is easy to pinpoint the error. Suppose f is meromorphic on $\mathbf{D}^*$ and has an essential singularity at the origin. Set $F_\theta(z) = f(z)\bar{f}(\bar{z}e^{i\theta})$. Lehto claims that $F_\theta(z)$ has an essential singularity at the origin for all $\theta \in [0, 2\pi)$ with at most one exception. The following example shows this is false. Set

$$f(z) = \prod_{n=1}^{\infty} \left[1 - \left(\frac{1}{nz} \right)^{3^n} \right] \bigg/ \prod_{n=1}^{\infty} \left[1 + \left(\frac{1}{nz} \right)^{3^n} \right].$$

Then f is meromorphic on $\mathbf{P} \setminus \{0\}$ and has an essential singularity at the origin. If $\theta = \pi j/3^m$, for j an odd integer and m a positive integer, then $F_\theta(z)$ is a rational function. Thus, $F_\theta(z)$ does not have an essential singularity at the origin for a countable, dense set of values in $[0, 2\pi)$. For $\theta \neq \pi j/3^m$ the function $F_\theta(z)$ does have an essential singularity at the origin. I want to thank David Styer for a helpful discussion that led to this example.

Lehto has informed me that he has been aware of the error regarding the singularity of $F_\theta(z)$ at the origin. In fact, a student of his rectified the error in her master's thesis. The paper was written in Finnish and was never published. Independently, I discovered a way to correct the error. It is presented below for the convenience of the reader.

In order that Lehto's proof be valid, all that is required is a single value of $\theta \in [0, 2\pi)$ with the property that $F_\theta(z)$ has an essential singularity at the origin. It is possible to show that $F_\theta(z)$ has a pole or removable singularity at the origin for at most countably many values of θ . My proof of this is tedious. For this reason the following slight modification of Lehto's proof seems preferable.

Assume f is meromorphic in $\mathbf{D}^*$ and has an essential singularity at the origin. Suppose there is a sequence $(z_n)_{n=1}^\infty$ in $\mathbf{D}^*$ with $z_n \rightarrow 0$ and $f(z_n) = 0$ for all n . Because f has just countably many poles in $\mathbf{D}^*$, it is possible to select θ so that $\bar{z}_n e^{i\theta}$ is not a pole of f for all n . Fix such a value of θ . Then $F_\theta(z_n) = 0$ for all n , so $F_\theta(z)$ has an essential singularity at the origin. Now, Lehto's proof allows us to conclude

$$\limsup_{z \rightarrow 0} |z|f^\#(z) \geq 1/2.$$

Of course, such a sequence $(z_n)_{n=1}^\infty$ need not exist. However, the big Picard theorem implies that for any $a \in \mathbf{P}$, with at most two exceptions, there is such a sequence $(z_n)_{n=1}^\infty$ with $f(z_n) = a$ for all n . Fix such a value a . Then $R(w) = (w - a)/(1 + \bar{a}w)$

is a rotation of $\mathbf{P}$ and $g = R \circ f$ is meromorphic on $\mathbf{D}^*$. Clearly, $g(z_n) = 0$ for all n , so the first part of the proof shows the desired result holds for g in place of f . Since the spherical derivative is invariant under rotations of the sphere, the theorem is established in all cases.

ADDED IN PROOF. The following recent paper considers lower bounds for Bloch constants for meromorphic functions defined on $\mathbf{C}$: I. B. Oshkin, *On schlicht disks of meromorphic functions*, Moscow Univ. Math. Bull. **38** (1983), 25–27.

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