ALEXANDER MODULES

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ABSTRACT. The Alexander modules of a link are the homology groups of the universal abelian cover of the complement of the link. For a link of n -spheres in S^{n+2} , we show that, if $n \geq 2$, the Alexander modules $A_2, \dots, A_n$ and the torsion submodule of A_1 are all of type L . This leads to a characterization, below the middle dimension, of the polynomial invariants of the link. These results were previously proven for the special case of boundary links.

The purpose of this note is to give a simplified proof of a generalization of one of the principal results of [Sa]. In that work, we showed that the *Alexander modules* of a higher-dimensional *boundary link* are of a certain kind, called *type L*. As a corollary, we were able to characterize the polynomial invariants of a higher-dimensional boundary link, below the middle dimension. Here, we prove the results for *all* higher-dimensional links.

Some of the results in this paper can be obtained from work of Dwyer [Dw] or Smith [Sm]. Lemma 1 is a direct consequence of a result of Lichtenbaum [Li]. However, we will give a unified treatment via finite free resolutions [BE, BE2].

The main result of this paper was claimed in [Sa] in a remark. We apologize for the delay.

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1. Modules of type L . Let Λ be the integral group ring of the free abelian group of rank m . We may identify Λ with the ring $\mathbf{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_m^{\pm 1}]$ of Laurent polynomials. Thus, Λ is a commutative, noetherian domain with global (homological) dimension $m + 1$, over which all finitely generated projective modules are free (see [Sw]). If M is a finitely generated Λ -module, we say that M is of *type L* if $\mathbf{Z} \otimes_{\Lambda} M = 0$. Here, $\mathbf{Z}$ is regarded as a Λ -module via the augmentation homomorphism $\epsilon: \Lambda \rightarrow \mathbf{Z}$.

We now state the main algebraic result of this paper. Let

$$\mathbf{F}: F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$$

be a complex of free Λ -modules of finite rank, and denote by $\mathbf{Z} \otimes \mathbf{F}$ the complex of free abelian groups obtained by tensoring the complex $\mathbf{F}$ with $\mathbf{Z}$ over Λ .

THEOREM A. *Suppose that $H_1(\mathbf{Z} \otimes \mathbf{F}) = 0$. Then $H_1(\mathbf{F})$ is of type L .*

The proof of this result depends on a small portion of the theory of finite free resolutions (see [BE, BE2]). For the convenience of the reader, we will summarize the aspects we need.

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Let R be a commutative noetherian ring with unit, and let $f: G \rightarrow H$ be a homomorphism of finitely generated free modules. A choice of bases for G and H allows us to associate a matrix to f . For each positive integer i , let U_i be the ideal in R generated by the $i \times i$ minor determinants of this matrix. We set $U_0 = R$, and for i sufficiently large, $U_i = 0$ because there are no $i \times i$ minors. The U_i form a decreasing sequence. The rank of f , denoted $\text{rk}(f)$, is the largest integer k for which $U_k \neq \{0\}$. For $k = \text{rk}(f)$, we set $I(f) = U_k$. The sequence U_i depends only on f .

The depth of an ideal J of R is the length of any maximal R -sequence formed from elements of J . For $J \neq R$, this is a well-defined integer. If $J = R$, we say $\text{depth}(J) = \infty$. The radical of J , denoted $\text{Rad}(J)$, is the set of all x such that some power of x is in J . We will use the following facts. Let

$$\mathbf{F}: 0 \rightarrow F_n \xrightarrow{\phi_n} F_{n-1} \rightarrow \cdots \rightarrow F_1 \xrightarrow{\phi_1} F_0$$

be a complex of free R -modules of finite rank.

(I) [BE, Corollary 1] $\mathbf{F}$ is exact if and if for each k , $\text{rk}(\phi_k) + \text{rk}(\phi_{k+1}) = \text{rk}(F_k)$ and $\text{depth}(I(\phi_k)) \geq k$.

(II) [BE2, Theorem 2.1] If $\mathbf{F}$ is exact, then for all k , $\text{Rad}(I(\phi_k)) \subseteq \text{Rad}(I(\phi_{k+1}))$.

Note that if $R = \mathbf{Z}$ in (I), the depth condition translates to the requirement that $I(\phi_k) = \mathbf{Z}$ for all $k > 1$.

LEMMA 1. *Let M be a finitely generated Λ -module such that $\text{Tor}_k^\Lambda(\mathbf{Z}, M) = 0$ for some $k \geq 0$. Then $\text{Tor}_j^\Lambda(\mathbf{Z}, M) = 0$ for all $j \geq k$.*

REMARKS. The case $k = 0$ was proved independently in [Dw, Sa and Sm]. This lemma is also a consequence of [Li].

PROOF. Choose a finite free resolution for M . This is a resolution of finite length $\{F_k, \phi_k\}$ such that each F_k is free of finite rank. Such a resolution always exists, since Λ is noetherian of finite global dimension and all finitely generated projectives are free [Sw]. If $\text{Tor}_k^\Lambda(\mathbf{Z}, M) = 0$, then $\mathbf{Z} \otimes_\Lambda F_{k+1} \rightarrow \mathbf{Z} \otimes_\Lambda F_k \rightarrow \mathbf{Z} \otimes_\Lambda F_{k-1}$ is exact (if $k = 0$, $F_{k-1} = 0$). By (I) above, $I(1 \otimes \phi_{k+1}) = \mathbf{Z}$. Since ϵ is onto, $\epsilon(I(\phi_{k+1})) = I(1 \otimes \phi_{k+1})$. However, by (II), $\text{Rad}(\phi_i)$ form an increasing sequence; hence $\epsilon(I(\phi_j)) = I(1 \otimes \phi_j) = \mathbf{Z}$ for all $j > k$. This implies that $\text{rk}(1 \otimes \phi_j) = \text{rk}(\phi_j)$ for all $j > k$. We now apply (I) to conclude that $\text{Tor}_j^\Lambda(\mathbf{Z}, M) = 0$ for all $j > k$. $\square$

PROOF OF THEOREM A. There is a universal coefficient spectral sequence for homology (see, for example, [HW]) with E^2 -term given by $E_{p,q}^2 = \text{Tor}_p^\Lambda(\mathbf{Z}, H_q(\mathbf{F}))$ and differential of degree $(-r, r-1)$ converging to $H_*(\mathbf{Z} \otimes \mathbf{F})$. Since $H_1(\mathbf{Z} \otimes \mathbf{F}) = 0$ by hypothesis, we must have $E_{0,1}^\infty = E_{1,0}^\infty = 0$. However, $E_{1,0}^\infty = E_{1,0}^2 = \text{Tor}_1^\Lambda(\mathbf{Z}, H_0(\mathbf{F}))$ and $E_{0,1}^\infty$ is the cokernel of $d_{2,0}^2: \text{Tor}_2^\Lambda(\mathbf{Z}, H_0(\mathbf{F})) \rightarrow \mathbf{Z} \otimes_\Lambda H_1(\mathbf{F})$. Since $\text{Tor}_1^\Lambda(\mathbf{Z}, H_0(\mathbf{F})) = 0$, we can apply Lemma 1 to see that $\text{Tor}_2^\Lambda(\mathbf{Z}, H_0(\mathbf{F})) = 0$. But then $d_{2,0}^2$ is a homomorphism of 0 onto $\mathbf{Z} \otimes_\Lambda H_1(\mathbf{F})$. Hence, $H_1(\mathbf{F})$ is of type L . $\square$

2. Alexander modules. Let X be the complement of an m -component link of n -spheres in S^{n+2} , $n \geq 2$. By Alexander duality, $H_1(X) \approx \mathbf{Z}^m$. If $\tilde{X}$ is the universal abelian cover of X , then the homology groups of $\tilde{X}$ are finitely generated Λ -modules. These modules are called the *Alexander modules* of the link.

THEOREM B. *Let $\tilde{X}$ be the universal abelian cover of the complement X of a link of n -spheres in S^{n+2} , $n \geq 2$. Then:*

- (i) $H_q(\tilde{X})$ is a Λ -module of type L for $2 \leq q \leq n$.
- (ii) If T_1 is the Λ -torsion submodule of $H_1(\tilde{X})$, then T_1 is of type L .

REMARK. We may now use results of [Sa] to completely characterize the polynomial invariants of a higher-dimensional link, below the middle dimension (see [Sa] for this characterization).

Part (i) of Theorem B follows directly from Theorem A, since $H_q(X) = 0$ for $2 \leq q \leq n$. Part (ii) requires a more circuitous approach.

Let F_m denote the free (nonabelian) group on m letters, and let $\pi = \pi_1(X)$. There is a homomorphism of F_m to π which carries the generators to meridians of the link. It follows from results of Kervaire [Ke] and Stallings [St] that this is a monomorphism, which induces an isomorphism on H_1 . Let V be the m -fold wedge of circles. The homomorphism $F_m \rightarrow \pi$ is realized by a mapping of V into X , which we can regard as an inclusion. Since the inclusion induces an identification on H_1 , we obtain an equivariant map $i: \tilde{V} \rightarrow \tilde{X}$. Thus, $i_*: H_*(\tilde{V}) \rightarrow H_*(\tilde{X})$ is a homomorphism of Λ -modules.

LEMMA 2. i_* induces an isomorphism

$$1 \otimes i_*: \mathbf{Z} \otimes_{\Lambda} H_1(\tilde{V}) \rightarrow \mathbf{Z} \otimes_{\Lambda} H_1(\tilde{X}).$$

In particular, if $K = \text{cok } i_*$, then K is of type L .

PROOF. The equivariant map i induces a homomorphism of spectral sequences of the coverings $\tilde{V} \rightarrow V$ and $\tilde{X} \rightarrow X$. At the E^2 -level, we have a commuting diagram

$$\begin{array}{ccc} H_0(\mathbf{Z}^m; H_1(\tilde{V})) & \rightarrow & H_0(\mathbf{Z}^m; H_1(\tilde{X})) \\ \uparrow & & \uparrow \\ H_2^m(\mathbf{Z}; H_0(\tilde{V})) & \rightarrow & H_2^m(\mathbf{Z}; H_0(\tilde{X})) \end{array}$$

where the horizontal maps are induced by i and the vertical maps are the E^2 -differentials. The bottom map is clearly an isomorphism. The vertical maps are isomorphisms as well; this follows from the fact that $H_1(V) \approx H_1(X) \approx \mathbf{Z}^m$ and an argument similar to that given in the proof of Lemma 1. Hence, all four maps are isomorphisms. Since $H_0(\mathbf{Z}^m; M) = \mathbf{Z} \otimes_{\Lambda} M$ for any Λ -module M , the lemma follows. The statement about K follows from the right-exactness of tensor product. $\square$

Any module of type L is a Λ -torsion module. This is because $\mathbf{Z} \otimes_{\Lambda} M \approx M/\mathcal{A}M$, where $\mathcal{A} = \ker\{\epsilon: \Lambda \rightarrow \mathbf{Z}\}$ is the ideal generated by $\{x_1 - 1, \dots, x_m - 1\}$ in Λ . Nakayama's lemma applies to show that if $\mathbf{Z} \otimes_{\Lambda} M = 0$, then M is annihilated by an element of Λ of the form $1 + a$, where $a \in \mathcal{A}$.

LEMMA 3. $i_*: H_1(\tilde{V}) \rightarrow H_1(\tilde{X})$ is one-to-one.

PROOF. Since $H_1(\tilde{V})$ is a Λ -torsion-free module of rank $m - 1$ (see, for example, [Sa, Lemma 2.5]), it will suffice to show that

$$1 \otimes i_*: \Lambda_0 \otimes_{\Lambda} H_1(\tilde{V}) \rightarrow \Lambda_0 \otimes_{\Lambda} H_1(\tilde{X})$$

is one-to-one where Λ_0 is the quotient field of Λ , and to do this, it will suffice to show that $1 \otimes i_*$ is onto and $\text{rk}(H_1(\tilde{X})) = m - 1$. The first part is easy; $\text{cok } i_* = K$ is of type L and hence Λ -torsion so that $\Lambda_0 \otimes_{\Lambda} K = 0$. Now $H_1(\tilde{X})$ depends only on π , so it is enough to look at any complex P with $\pi_1(P) \approx \pi$. It is essentially a result of Kervaire [Ke, Theorem 3] that there exists a finite 3-complex P with only one 0-cell such that $\pi_1(P) \approx \pi$ and $H_*(P) \approx H_*(V)$. The Euler characteristic of such a

P is $1 - m$; hence, the Euler characteristic of $\tilde{P}$ over Λ is $1 - m$. By Theorem A, $H_2(\tilde{P})$ and $H_3(\tilde{P})$ are of type L and hence of rank zero. $H_0(\tilde{P}) \approx \mathbf{Z}$ and has rank zero. It follows that $H_1(\tilde{P}) \approx \pi'/\pi'' \approx H_1(\tilde{X})$ has rank $m - 1$. $\square$

We have, as a consequence of Lemma 3, a short exact sequence

$$0 \rightarrow H_1(\tilde{V}) \xrightarrow{i_*} H_1(\tilde{X}) \xrightarrow{j} K \rightarrow 0$$

and K is of type L . Let T_1 be the Λ -torsion submodule of $H_1(\tilde{X})$, and let F denote $i_*H_1(\tilde{V}) \subseteq H_1(\tilde{X})$. Since F is torsion-free, $T_1 \cap F = \{0\}$.

LEMMA 4. T_1 is of type L .

PROOF. It suffices to show that $j|T_1: T_1 \rightarrow K$ is one-to-one. For if this is so, let $C = \text{cok}(j|T_1)$. Then $\mathbf{Z} \otimes_{\Lambda} K = 0$ implies $\mathbf{Z} \otimes_{\Lambda} C = 0$. By Lemma 1, $\text{Tor}_1^{\Lambda}(\mathbf{Z}, C) = 0$. By the long exact sequence of Tor , we get that $\mathbf{Z} \otimes_{\Lambda} T_1 = 0$ because $\mathbf{Z} \otimes_{\Lambda} T_1$ falls between the two (zero) modules $\text{Tor}_1^{\Lambda}(\mathbf{Z}, C)$ and $\mathbf{Z} \otimes_{\Lambda} K$.

Suppose $t \in T_1$ such that $(j|T_1)(t) = 0$. Then $t = i_*(x)$ for some $x \in H_1(\tilde{V})$. Thus, $t \in T_1 \cap F = \{0\}$. Therefore, $t = 0$ and hence, $j|T_1$ is one-to-one. $\square$

This completes the proof of Theorem B. We may now use the characterization theorem for the polynomials of boundary links [Sa, Theorem 6.1] in the context of general links. The proof is the same.

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