A NOTE ON ŠČEPIN'S THEOREM

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ABSTRACT. We shall generalize a theorem of Ščepin as follows.

THEOREM. *Every finite dimensional $\text{ANR}(P_1)$ is metrizable, where P_1 denotes the class of all σ -locally compact, paracompact p -spaces.*

1. Introduction. In this paper we assume that all spaces are Tychonoff and mappings are continuous. For a space X , $\dim X$ denotes the covering dimension of X (cf. [1]). We use the notation $\text{ANR}(\mathcal{T})$ in the sense of [2] for some class of topological spaces $\mathcal{T}$. For the undefined terminology refer to [1 and 2].

Recently, Ščepin has proved the following.

THEOREM A [9]. *Every finite dimensional $\text{ANR}(\mathcal{C})$ is metrizable, where $\mathcal{C}$ denotes the class of all compact spaces.*

He has also generalized this result as follows. We say that a space X is a *generalized Peano continuum* if it is a continuous image of some Tychonoff cube I^τ .

THEOREM B [10]. *Every nonmetrizable generalized Peano continuum X contains a topological copy of Tychonoff cube I^μ for every regular cardinal $\mu \leq \underline{w}(X)$, where $\underline{w}(X)$ denotes the weight of the space X .*

REMARK 1. Observe that Theorem A is a consequence of Theorem B. This follows from the fact that every $\text{ANR}(\mathcal{C})$ is a finite sum of generalized Peano continua. Indeed, since every $\text{ANR}(\mathcal{C})$ is a topological sum of its connected components, it suffices to see that every connected $\text{ANR}(\mathcal{C})$ is a generalized Peano continuum. Let X be any connected $\text{ANR}(\mathcal{C})$. Then, we can embed it in I^τ , where $\tau = \underline{w}(X)$. Since X is an $\text{ANR}(\mathcal{C})$, we have a finite subset $A \subset \tau$ and an open set U in I^A such that there exists a retraction $r: \pi_A^{-1}(U) \rightarrow X$, where $\pi_A: I^\tau \rightarrow I^A$ is a natural projection. Since I^A is a finite dimensional cube and $\pi_A(X)$ is connected compact, we have a connected polyhedral neighborhood P of $\pi_A(X)$ in I^A such that $\pi_A(X) \subset \text{int } P \subset P \subset U$ after a suitable subdivision of a triangulation of I^A . Since P is a connected locally connected compact metric space, we have an onto map $s: I \rightarrow P$. Put

$$f = r \circ (\text{id}_{\tau-A} \times s),$$

where $\text{id}_\mu: I^\mu \rightarrow I^\mu$ is the identity mapping. Then, f is a continuous onto map from I^τ to X . Thus, X is a generalized Peano continuum.

The purpose of this paper is to generalize Theorem A in a different direction (a space is said to be σ -locally compact if it is a countable union of locally compact closed subspaces):

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THEOREM 1. *Every finite dimensional $\text{ANR}(\mathcal{P}_1)$ is metrizable, where $\mathcal{P}_1$ denotes the class of all σ -locally compact, paracompact p -spaces.*

REMARK 2. (a) Let $\mathcal{P}$ and $\mathcal{T}_0$ be the class of all paracompact p -spaces and the class of all spaces each of which has a σ -closure preserving cover by compact sets, respectively. Then, by a theorem of R. Telgársky [12] we see that $\mathcal{P}_1 = \mathcal{P} \cap \mathcal{T}_0$. Hence, the class $\mathcal{P}_1$ is finite productive, closed hereditary, and also has the following property.

(i) If $X \in \mathcal{P}_1$ and $f: X \rightarrow Y$ is a perfect map, then $Y \in \mathcal{P}_1$. (More generally, $Y \in \mathcal{P}_1$, when $X \in \mathcal{P}_1$, Y is a p -space, and $f: X \rightarrow Y$ is a closed map.)

(b) $\text{ANR}(\mathcal{P})$'s and spaces which have σ -closure preserving cover by compact sets have been found to have many interesting properties (cf. [3, 4, 7, 8, 11 and 12]).

Let $\mathcal{P}_0$ be the class of all locally compact paracompact spaces. Then, since every $\text{ANR}(\mathcal{P}_0)$ is an $\text{ANR}(\mathcal{P}_1)$, we have the following

COROLLARY 1. *Every finite dimensional $\text{ANR}(\mathcal{P}_0)$ is metrizable.*

To prove Theorem 1, we need the following theorem which is a slight improvement for the case $\text{ANR}(\mathcal{P}_1)$ of a theorem of Telgársky [12].

THEOREM 2. *Let X be an $\text{ANR}(\mathcal{P}_1)$. Then, X is covered by a σ -discrete collection of generalized Peano continua.*

A counterexample and some unsolved problems are given in §3 of this paper.

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2. Proofs of Theorems 1 and 2. We start with the proof of Theorem 2. By [5, Theorem 1], we can assume that X is the closed subset

$$\{(e(x), f(x)) \in I^\tau \times T : x \in X\}$$

of a product space $I^\tau \times T$ of a Tychonoff cube I^τ ($\tau = \omega(X)$) and a metric space T , where $e: X \rightarrow I^\tau$ is some Tychonoff embedding and $f: X \rightarrow T$ is some perfect onto mapping.

Now, since f is perfect and X is a member of $\mathcal{P}_1$, T and $I^\tau \times T$ are members of $\mathcal{P}_1$. Therefore, since X is an $\text{ANR}(\mathcal{P}_1)$, there exists a retraction $r: U \rightarrow X$ for some neighborhood U of X in $I^\tau \times T$. By [12, Theorem 13.1] we can put $X = \bigcup_{i=1}^\infty X_i$, where each X_i is a closed subset of X and is a topological sum of compact sets $C_{i,\lambda}$, $\lambda \in \Lambda_i$. By the collectionwise normality of X , we can choose a discrete collection $\mathcal{V}_i = \{V_{i,\lambda} : \lambda \in \Lambda_i\}$ of open subsets in X such that $V_{i,\lambda} \supset C_{i,\lambda}$. Since $C_{i,\lambda}$ is compact, there exists a finite subset $A \subset \tau$ and a compact metric space $P_{i,\lambda} \subset I^A \times T$ such that

$$(1) \quad r^{-1}(V_{i,\lambda}) \supset I^{\tau-A} \times P_{i,\lambda} \supset C_{i,\lambda}.$$

(Compare with Remark 1.) For each $P_{i,\lambda}$, we can choose a perfect onto mapping $h_{i,\lambda}: C \rightarrow P_{i,\lambda}$ from a Cantor-set C in a real line R . Put

$$g_{i,\lambda} = r \circ (\text{id}_\mu \times h_{i,\lambda}), \quad \text{where } \mu = \tau - A.$$

Then, since $V_{i,\lambda}$ is an $\text{ANE}(\mathcal{P}_1)$ (cf. [2, p. 42]), $g_{i,\lambda}$ has a neighborhood extension $\tilde{g}_{i,\lambda}: I^\mu \times U_{i,\lambda} \rightarrow V_{i,\lambda}$ for some open neighborhood $U_{i,\lambda}$ of C in the real line R . Put

$$U_{i,\lambda} = \bigcup_{k=1}^{\infty} I_k,$$

where each I_k is a closed interval, and put

$$G_{i,\lambda,k} = \tilde{g}_{i,\lambda}(I^\mu \times I_k).$$

Then, from (1) and the definition of $g_{i,\lambda}$,

$$V_{i,\lambda} \supset \bigcup_{k=1}^{\infty} G_{i,\lambda,k} \supset g_{i,\lambda}(I^\mu \times C) \supset r(I^\mu \times P_{i,\lambda}) \supset r(C_{i,\lambda}) = C_{i,\lambda}.$$

For each pair of integers i and k , put

$$\mathcal{G}_{i,k} = \{G_{i,\lambda,k} : \lambda \in \Lambda_i\}.$$

Then, since $\mathcal{V}_i$ is a discrete collection, $\mathcal{G}_{i,k}$ is a discrete collection of generalized Peano continua. Hence, $\mathcal{G} = \bigcup_{i,k=1}^{\infty} \mathcal{G}_{i,k}$ is the required covering consisting of generalized Peano continua.

PROOF OF THEOREM 1. Let X be a nonmetrizable $\text{ANR}(\mathcal{P}_1)$. Then, by Theorem 2, X is covered by a σ -discrete collection $\mathcal{G}$ of generalized Peano continua. If each member G of $\mathcal{G}$ is metrizable, we can show that X has a σ -discrete network, using countable bases of all G 's. Thus, X is a paracompact p -space. Then, X is metrizable, since every paracompact p -space with a σ -discrete network is metrizable [6, Theorem 3.6], which leads to a contradiction. Therefore, there exists a nonmetrizable generalized Peano continuum G in $\mathcal{G}$. Then, by Theorem B in §1, G contains a topological copy of I^{ω_1} , where ω_1 is the first uncountable cardinal. Hence, X also contains a topological copy of I^{ω_1} and therefore is infinite dimensional.

3. An example and some unsolved problems. The following example shows that we *cannot* expect that the space X in our theorems contains a copy of I^μ for every regular cardinal $\mu \leq \underline{w}(X)$.

EXAMPLE. For every pair of cardinals $\alpha < \beta$ there exists a connected $\text{ANR}(\mathcal{P}_1)$ space $X_{\alpha,\beta}$ with $\underline{w}(X_{\alpha,\beta}) = \beta$ such that (i) $X_{\alpha,\beta}$ contains a topological copy of I^α , but does not contain any topological copy of I^γ for any $\gamma > \alpha$.

Let $J(\beta)$ be the hedgehog space of size β , and put $X_{\alpha,\beta} = I^\alpha \times J(\beta)$. Then, $X_{\alpha,\beta}$ is an $\text{ANR}(\mathcal{P}_1)$ (cf. [2, 8]). One can show without difficulty that $X_{\alpha,\beta}$ is the required one.

The following problem is one of the most interesting unsolved ones in this direction.

Problem 1 (*R. Telgársky* [13]). Is every finite dimensional $\text{ANR}(\mathcal{P})$ metrizable (cf. Remark 2(a))?

In particular, the following problem remains open.

Problem 2. Let X be a finite dimensional retract of $I^{\omega_1} \times R^\omega$. Then, is X metrizable?

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