SOME INTERSECTION PROPERTIES OF THE FIBRES OF SPRINGER'S RESOLUTION

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ABSTRACT. Combinatorial results are used to calculate the dimension of the intersection of any two irreducible components of the set in the flag variety fixed by the action of a unipotent element of GL_n whose Jordan decomposition has two blocks. This is then related to the "left cells" of Kazhdan and Lusztig, which are used to construct representations of S_n , the Weyl group of GL_n .

0. In this note, we study the fibres of Springer's resolution [St] of the singularities of the unipotent variety in $G = GL_n(k)$, where k is an algebraically closed field. These fibres are fixed point sets for the action of G on the variety $\mathcal{B}$ parametrizing the complete flags in a vector space of dimension n . We use $\mathcal{B}_u$ to fixed by a unipotent element $u \in SL_n$.

In general, $\mathcal{B}_u$ has several irreducible components. Suppose that the Jordan decomposition of u has block sizes $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$, where $\lambda_1 + \cdots + \lambda_s = n$. (We refer to this as the *shape* λ , and, by abuse of language, we say that u has *shape* λ .) Then each component of $\mathcal{B}_u$ has dimension $\sum_{i=1}^s (i-1)\lambda_i$, and there is one component for each standard Young tableau of shape λ (see [S] and below). However, one does not know, in general, the codimension of the intersection of two components; only the "one-hook" case has been done [V]. Here, we calculate the codimension when the Jordan form of u has two blocks. This calculation depends on combinatorial techniques exposed in [LS]. The precise result is (2.1).

This calculation has two-fold significance. First, it enables one, in this case, to verify a conjecture of Kazhdan and Lusztig [KL, 6.3] concerning the configuration of components of $\mathcal{B}_u$; see (4.3). Second, this casts new light on the combinatorial results in [LS] which are used to calculate Kazhdan-Lusztig polynomials, because there is no mention of the geometry of the Grassmannian here.

1.0. In this section, we begin to associate (following [LS]) components of $\mathcal{B}_u$ (for certain u) to words in the letters α and β (and vice-versa). Let M_a^b denote the set of words made up of a α 's and b β 's, e.g., $M_2^1 = \{\beta\alpha\alpha, \alpha\beta\alpha, \alpha\alpha\beta\}$. We associate a permutation of $\{1, 2, \dots, a+b\}$ to each $w \in M_a^b$ by writing $1, \dots, a$ in order from right to left under the α 's, and then $a+1, \dots, a+b$ in order from right to left under the β 's. Thus, $\alpha\alpha\beta\alpha\beta\beta\alpha$ is associated to 4, 3, 7, 2, 6, 5, 1, which can be written (14237)(56).

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(1.1) Suppose that $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s)$ is a shape. A *tableau* of shape λ consists of s top-justified columns of respective sizes $\lambda_1, \dots, \lambda_s$. Thus $(4 \geq 3 \geq 1 \geq 1)$ corresponds to

A *standard Young tableau* of shape λ is a tableau filled with the integers $1, \dots, \Sigma \lambda_i$ such that the rows decrease from left to right and the columns decrease from top to bottom (this is not the usual convention). Thus, the standard Young tableaux of shape $3 \geq 2$ are

(1.2) There is a one-to-one correspondence between pairs of Young tableaux with n boxes and elements of S_n , the symmetric group; this is the *Robinson-Schensted correspondence* (see, e.g. [Kn]). We use the following nonstandard insertion algorithm: let $t_1, \dots, t_s$ be the top row of a tableau T , and let T' be the remainder of T . We insert l as follows: if $l > t_1$, the resulting tableau has top row $l, t_2, \dots, t_s$, and t_1 is inserted into T' . If $t_i > l > t_{i+1}$, the top row of the resulting tableau is $t_1, \dots, t_i, l, t_{i+2}, \dots, t_s$ and t_{i+1} is inserted into T' . (It is convenient to assume $-\infty$ is at the end of each row.) To create a pair of tableaux corresponding to $\sigma \in S_n$, insert $\sigma(n), \sigma(n-1), \dots, \sigma(1)$ in the left tableau while filling the right tableau with $n, n-1, \dots, 1$ as each new cell is created. To construct a permutation from a tableau pair, exchange the left and right tableaux and imagine how each element was inserted, starting with the right tableau entry whose place is occupied by 1 in the left tableau.

(1.3) EXAMPLE. We construct the tableau pair for $\alpha\alpha\beta\alpha\beta$. The permutation σ has the following action: $\sigma(1) = 3, \sigma(2) = 2, \sigma(3) = 5, \sigma(4) = 1, \sigma(5) = 4$. We insert 4, 1, 5, 2 and 3 into a tableau, with the result:

Conversely, take the pair

$$\begin{array}{cc} 5 & 4 \\ 3 & 2 \\ 1 & \end{array} \quad \begin{array}{cc} 5 & 3 \\ 4 & 2 \\ 1 & \end{array}$$

The 1 on the right was “bumped” by 2, which was bumped by 3, so $\sigma(1) = 3$, etc.

(1.4) PROPOSITION. *The tableaux associated to $w \in M_a^b$ have 2 columns, the second of which has length $\leq b$.*

PROOF. Notice that the longest increasing subsequence in the permutation associated to w has length 2. The result now follows from [Kn, 5.1.4, exercise 7]. $\square$

(1.5) Let $u \in \text{SL}_n$ be a unipotent of shape λ . Following [S], we set up a one-to-one correspondence between components of $\mathcal{B}_u$ and Young tableaux of shape λ . Let e be the $n \times n$ identity, and write $\tilde{u} = u - e$. Suppose T is a tableaux of shape λ such that 1 is in row ρ . Then the generic flag $F_1 \subset F_2 \subset \cdots \subset F_n$ in the component associated to T satisfies

$$F_1 \subset \ker(\tilde{u}) \cap [\text{image}(\tilde{u}^{\rho-1}) \setminus \text{image}(\tilde{u}^\rho)];$$

$F_2, \dots, F_n$ are determined by following this procedure inductively.

(1.6) It is now clear how $w \in M_a^b$ corresponds to a component of $\mathcal{B}_u$, where u has shape $\lambda_1 \geq \lambda_2$ with $\lambda_1 \geq a$, $\lambda_2 \leq b$. Assume that the first letter of w is α (if not, the second columns of the tableaux associated to w have length $< b$, since n is the last integer inserted); then w corresponds to a pair of tableaux (T_L, T_R) of shape $a \geq b$. Take the tableau T_R ; it corresponds to a component of $\mathcal{B}_u$. Conversely, a component of $\mathcal{B}_u$ corresponds to a tableau T_R , and (T_L, T_R) corresponds to some $w \in M_a^b$ (that this is unique follows from the proof of (1.4)).

(1.7) We label each $w \in M_a^b$ with the integer j if the j th, $(j+1)$ st letters of w are $\alpha\beta$; thus $\alpha\alpha\beta\alpha\beta$ has the labels 2 and 4.

REMARK. The labels of w are clearly the left descent set of w when S_n is thought of as a Coxeter group of type A_{n-1} .

PROPOSITION. *The labels of w are the parabolic lines contained in the component of $\mathcal{B}_u$ associated to w .*

PROOF. Recall that a component of $\mathcal{B}_u$ contains parabolic lines of type j if one is free to choose F_j such that $F_{j-1} \subset F_j \subset F_{j+1}$ when F_{j-1}, F_{j+1} are fixed in $F_1 \subset \cdots \subset F_n$. It is clear from (1.5) that this is the case exactly when j is above or at the same level as $j+1$ in the tableau associated to the component.

Suppose w has the label j , and that σ is the associated permutation. Then $\sigma(j+1) > \sigma(k)$ for $k > j+1$, since there is a β in the $(j+1)$ st spot and β 's increase from right to left. Thus, the new cell in the tableau is at the bottom of the first column, since $\sigma(j+1)$ “bumps” the element in the upper left-hand corner. However, $\sigma(j) < \sigma(j+1)$, since all α are less than all β , so the new cell for $\sigma(j)$ is at the end of column 2. Since column 2 is always shorter than column 1 (or equal in length), the result follows. $\square$

This can often be used to associate a word to a tableau quickly, since e.g. the tableau

$$\begin{array}{cc} 5 & 4 \\ 3 & 2 \\ 1 & \end{array}$$

indicates lines of type 2 and 4, hence labels 2 and 4, hence $\alpha\alpha\beta\alpha\beta$.

(1.8) PROPOSITION. Suppose $w \in M_a^b$ begins with $\alpha\beta\beta$. Then w does not correspond to a component of $\mathcal{B}_u$ where u has shape $a \geq b$.

PROOF. Suppose the contrary; then the second column of the associated tableau has length b . Write $w = \alpha\beta\beta w'$, where $w' \in M_{a-1}^{b-2}$; the second columns of the w' tableaux have length $\leq b-2$. Insertion of $a+b-1$, $a+b$, and a into these tableaux give the w tableaux, but the first 2 elements go into the first column, so the second column has length $\leq b-1$. This is a contradiction. $\square$

2.0. A pair $(v, w) \in M_a^b \times M_a^b$ is *reduced* if all coincident phrases $\alpha\beta$ are removed, e.g. if $(v, w) = (\alpha\alpha\beta\alpha\beta, \alpha\alpha\beta\beta\alpha)$, the reduced pair is $(\alpha\alpha\beta, \alpha\beta\alpha)$. This has the effect of removing common labels.

Each $w \in M_a^b$ is associated to a path in the plane $\mathbb{R}^2$ by starting at $(0, 0)$ and assigning segments $(m, n) \rightarrow (m+1, n-1)$ to each α and $(m, n) \rightarrow (m+1, n+1)$ to each β , e.g. $\alpha\alpha\beta\alpha\beta$:

We can now state our result.

(2.1) THEOREM. Associate $v, w \in M_a^b$ to components $X_v, X_w \subset \mathcal{B}_u$, where u has shape $a \geq b$.

(a) Suppose the reduced form of (v, w) is

$$v^{\text{red}} = w_0 x_1 w_1 x_2 w_2 \cdots x_m w_m, \quad w^{\text{red}} = w_0 \tilde{x}_1 w_1 \tilde{x}_2 w_2 \cdots \tilde{x}_m w_m,$$

where $x_i \in M_1^1$ and $\tilde{\alpha}\beta = \beta\alpha$, $\tilde{\beta}\alpha = \alpha\beta$. Then

$$\text{codim}_{X_v}(X_v \cap X_w) = m.$$

(b) If v^{red} and w^{red} do not have this form, then the components do not meet.

REMARK. The codimension is the number of “diamonds” when the paths of (2.0) are superimposed, e.g.,

Codimension is 2.

3.0.

(3.1) We prove (2.1) by induction on b . When $b = 1$, the components have dimension 1, and may be identified by their labels, as each is a parabolic line. Parabolic lines of type i meet those of type j when $j = i + 1$ or $i = j + 1$. The words are

$$\begin{aligned} \alpha \cdots \alpha \beta \alpha \cdots \alpha & \quad (\beta \text{ in } (i+1)\text{st spot}), \\ \alpha \cdots \alpha \alpha \beta \alpha \cdots \alpha & \quad (\beta \text{ in } (i+2)\text{nd spot}). \end{aligned}$$

The conclusion follows.

This also proves (2.1)(b).

(3.2) Notice that the removal of a label reduces the component dimension by 1 (by (1.8), no word begins $\alpha\beta\beta$). If components X_1, X_2 both contain parabolic lines of type j , and if $\mathcal{F} = (F_1 \subset F_2 \subset \cdots \subset F_{j-1} \subset F_j \subset F_{j+1} \subset \cdots \subset F_n)$ is a flag in $X_1 \cap X_2$, then so is $F_1 \subset \cdots \subset F_{j-1} \subset F'_j \subset F_{j+1} \subset \cdots \subset F_n$. Thus, the existence of a common label adds 1 to the dimension of each component and 1 to the dimension of the intersection, so the codimension is unchanged. Thus, we can assume that v and w are reduced.

(3.3) Suppose that

$$v = v_0 x_1 v_1 \cdots x_m v_m, \quad w = v_0 \tilde{x}_1 v_1 \cdots \tilde{x}_m v_m.$$

The word v_m has the form $\beta \cdots \beta \alpha \cdots \alpha$, since it has no label. Suppose it has p β 's and q α 's; then by (1.7), the subspaces $F_{n-p-q} \subset \cdots \subset F_n$ of a general flag in each component are forced, and we can reduce to the case of a unipotent of shape $a - p \leq b - q$ acting on F_{n-p-q} ; the result is known here by induction.

Thus, renaming if necessary, we are reduced to the case

$$v = v_0 x_1 v_1 \cdots v_{m-1} \alpha \beta, \quad w = v_0 \tilde{x}_1 v_1 \cdots v_{m-1} \beta \alpha.$$

The tableau describing X_v is

n	$n - 1$
$\vdots$	$\vdots$

and that for X_w is

n	
$n - 1$	$\vdots$
$\vdots$	

Choose $\mathcal{F} = (F_1 \subset \cdots \subset F_n)$ in the intersection. F_n is the space on which u acts, while F_{n-1} is forced by the tableau for w . However, there is a $\mathbf{P}^1$ of choices for F_{n-1} in the component for v , since it contains parabolic lines of type $n - 1$. Thus, we lose 1 dimension of choice in X_w , so

$$\text{codim}_{X_w}(X_v \cap X_w) = 1 + \text{codim}_{X_w}(X_{\tilde{v}} \cap X_{\tilde{w}}),$$

where

$$\tilde{v} = v_0 x_1 v_1 \cdots v_{m-1}, \quad \tilde{w} = v_0 \tilde{x}_1 v_1 \cdots v_{m-1}.$$

This induction step completes the proof. $\square$

4.0. Here is an application of (2.1). Let $l: S_n \rightarrow \mathbf{Z}$ be the length function. For each $y, w \in S_n$, there is a *Kazhdan-Lusztig polynomial* $P_{y,w} \in \mathbf{Z}[q]$ whose degree does not exceed $\frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$ in the Bruhat order. Let $\mu(y, w)$ be the coefficient of the term of that degree in $P_{y,w}(q)$, and, following [KL, §1], define an “ S_n -graph” as follows: the nodes are elements of S_n , and elements y and w are connected by an edge when either $\mu(y, w)$ or $\mu(w, y)$ is nonzero. Further, each node is labeled with its left descent set.

(4.1) Define another family of graphs as follows: let $u \in \text{SL}_n$ be a unipotent. The nodes are components of $\mathcal{B}_u$, and two such are connected if the components meet in codimension 1. The nodes are labelled with the set of parabolic lines contained in the associated component.

(4.2) Suppose that $y_1, y_2 \in S_n$ are permutations constructed, as in (1.0), from w_1, w_2 .

THEOREM [LS]. *Suppose the reduced forms of w_i are*

$$w_1 = v_0 x_1 v_1 \cdots x_m v_m, \quad w_2 = v_0 \tilde{x}_1 v_1 \cdots \tilde{x}_m v_m,$$

with $x_i \in M_1^1$. Then $\mu(y_1, y_2) = 1$ when $m = 1$; otherwise $\mu(y_1, y_2) = 0$.

(4.3) In [KL, 6.3] it is suggested that each graph in (4.1) should be a “ W -graph”, as in (4.0). It is an immediate consequence of (2.1) and (4.2) that this is the case when u is a unipotent with shape $a \geq b$. In particular, the intersection graph of $\mathcal{B}_u$ will be exactly the “left cell” containing the involution corresponding to $w = \alpha \cdots \alpha \beta \cdots \beta$, where w has a α ’s and b β ’s (this is an involution since it is a product of the longest elements of S_a acting on $\{1, \dots, a\}$ and S_b acting on $\{a+1, \dots, a+b\}$). See (1.7) to verify that the labelling of the nodes is correct.

REFERENCES

- [KL] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.
- [Kn] D. E. Knuth, *The art of computer programming*, Vol. III, Addison-Wesley, Reading, Mass., 1973.
- [LS] Alain Lascoux and M-P. Schützenberger, *Polynômes de Kazhdan and Lusztig pour les Grassmanniennes*, Astérisque, Soc. Math. France, Paris, **87–88**.
- [S] N. Spaltenstein, *The fixed point set of a unipotent transformation on the flag manifold*, Nederl. Akad. Wetensch. Proc. Ser. A **79** = Indag. Math. **38** (1976), 452–456.
- [St] R. Steinberg, *Conjugacy classes in algebraic groups*, Lecture Notes in Math., vol. 366, Springer-Verlag, New York, 1974.
- [V] J. A. Vargas, *Fixed points under the action of unipotent elements of SL_n* , Bol. Soc. Mat. Mexicana (2) **24** (1979), 1–14.

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