AN INTEGRAL MEAN ESTIMATE FOR POLYNOMIALS

M. A. MALIK¹

ABSTRACT. An inequality involving

$$\max_{|z|=1} |P'(z)| \quad \text{and} \quad \left(\int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right)^{1/q}, \quad q > 0,$$

for a polynomial $P(z)$ having all its zeros in $|z| \leq 1$ is presented.

1. Let $P(z)$ be a polynomial of degree n and $P'(z)$ its derivative. Concerning the estimate of $|P'(z)|$ when there is a restriction on the zeros of $P(z)$, Turan [9] proved the following:

THEOREM A. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then*

$$(1) \quad \frac{n}{2} \max_{|z|=1} |P(z)| \leq \max_{|z|=1} |P'(z)|.$$

The result is best possible and equality in (1) holds for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$.

In this paper we present a generalization of Theorem A in the sense that the left-hand side of (1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$. In fact, our result is related to the following one of Saff and Sheil-Small [8]:

THEOREM B. *If $P(z)$ is a polynomial of degree n having all its zeros on $|z| = 1$, then for each $q > 0$*

$$(2) \quad \left(\int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right)^{1/q} \leq (A_q)^{1/q} \left(\max_{|z|=1} |P(z)|/2 \right)$$

where

$$A_q = 2^{q+1} \sqrt{\pi} \Gamma\left(\frac{1}{2}q + \frac{1}{2}\right) / \Gamma\left(\frac{1}{2}q + 1\right).$$

The result is best possible and equality in (2) holds if and only if $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$.

In the proof of Theorem B, Saff and Sheil-Small established the inequality

$$(3) \quad n \left(\int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right)^{1/q} \leq (A_q)^{1/q} \max_{|z|=1} |P'(z)|$$

for the polynomials $P(z)$ having all its zeros on $|z| = 1$, and then in (3) used the Erdős-Lax theorem [3] to arrive at (2).

Received by the editors March 23, 1982.

1980 *Mathematics Subject Classification.* Primary 30A10; Secondary 30C10.

Key words and phrases. Polynomials, derivative, zeros, inequalities.

¹Research supported by NSERC, Canada.

©1984 American Mathematical Society
0002-9939/84 \$1.00 + \$.25 per page

2. We now present our main result:

THEOREM 1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for each $q > 0$*

$$(4) \quad n \left(\int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right)^{1/q} \leq (A_q)^{1/q} \max_{|z|=1} |P'(z)|$$

where $A_q = 2^{q+1} \sqrt{\pi} \Gamma(\frac{1}{2}q + \frac{1}{2}) / \Gamma(\frac{1}{2}q + 1)$. The result is best possible and equality in (4) holds for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$.

REMARK 1. For a polynomial $P(z)$ having all its zeros in $|z| > 1$, an inequality like (4) is not expected because the right-hand side of (4) may become arbitrarily small while $|P(z)|$ is uniformly close to 1 on $|z| = 1$. This follows from a result due to MacLane [4]: for any $\varepsilon > 0$ there exists a sequence of polynomials $T_k(z)$ having all its zeros on $|z| = 1 + \varepsilon$ and $T_k(z) \rightarrow 1$ uniformly on $|z| \leq 1 + \varepsilon'$ as $k \rightarrow \infty$, $\varepsilon' < \varepsilon$. This observation makes Theorem 1 interesting.

REMARK 2. Making $q \rightarrow \infty$ in (4), one gets (1).

PROOF OF THEOREM 1. Let $P(z)$ be a polynomial of degree n . Define $Q(z) = z^n \overline{P(1/\bar{z})}$. It is immediate that $|P(z)| = |Q(z)|$ for $|z| = 1$. Moreover, $P'(z) = z^n Q'(1/\bar{z})$ so

$$P'(z) = nz^{n-1} \overline{Q(1/\bar{z})} - z^{n-2} \overline{Q'(1/\bar{z})},$$

from which

$$z^{n-1} \overline{P'(1/\bar{z})} = nQ(z) - zQ'(z).$$

Since $|z^{n-1} \overline{P'(1/\bar{z})}| = |P'(z)|$ for $|z| = 1$, it further implies that

$$(5) \quad |P'(z)| = |nQ(z) - zQ'(z)|.$$

We also have

$$(6) \quad n|Q(z)| = |nQ(z) - zQ'(z) + zQ'(z)|$$

$$(7) \quad = |nQ(z) - zQ'(z)| |1 + w(z)|$$

where $w(z) = zQ'(z)/(nQ(z) - zQ'(z))$. Hence for $|z| = 1$,

$$(8) \quad n|P(z)| = |P'(z)| |1 + w(z)|.$$

Next, we show that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then $w(z)$ is analytic in $|z| \leq 1$ and $1 + w(z)$ is subordinate to $1 + z$ for $|z| \leq 1$.

Since $P(z)$ has all its zeros in $|z| \leq 1$, by the Gauss-Lucas theorem $P'(z)$ also has all its zeros in $|z| \leq 1$. This implies $z^{n-1} \overline{P'(1/\bar{z})} = nQ(z) - zQ'(z) \neq 0$ in $|z| < 1$. Moreover, if $|\alpha| = 1$ is a zero of order k ($< n$) of $nQ(z) - zQ'(z)$, then from (5) it is also a zero of order k of $P'(z)$. As the unit disk which contains all the zeros of $P(z)$ is strictly convex, following the Gauss-Lucas theorem, α is a zero of order $k+1$ of $P(z)$ and so of $Q(z)$. Hence α is a zero of order k of $Q'(z)$. This concludes that $w(z)$ is analytic in $|z| \leq 1$. To show that $|w(z)| \leq 1$ for $|z| = 1$ we note [5] that $|Q'(z)| \leq |P'(z)|$ for $|z| = 1$. Also $w(0) = 0$. Thus $1 + w(z)$ is subordinate to $1 + z$ for $|z| \leq 1$.

Applying a well-known property of subordination [2] to (8) we deduce that

$$\begin{aligned} n \left(\int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right)^{1/q} &\leq \max_{|z|=1} |P'(z)| \left(\int_0^{2\pi} |1 + w(e^{i\theta})|^q d\theta \right)^{1/q} \\ &\leq \max_{|z|=1} |P'(z)| (A_q)^{1/q}. \end{aligned}$$

This completes the proof of Theorem 1.

3. By using the techniques in the above proof, we prove

THEOREM 2. *If $P(z)$ is a polynomial of degree n , then*

$$(9) \quad \max_{|z|=1} (|P'(z)| + |Q'(z)|) = n \max_{|z|=1} |P(z)|$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$. Moreover, the maximums of both sides in (9) are attained at the same point $|z_0| = 1$.

PROOF OF THEOREM 2. Let $P(z)$ be a polynomial of degree n . It is known [5] that for $|z| \leq 1$

$$(10) \quad |P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)|.$$

We also note that from (5) and (6) one gets

$$(11) \quad \begin{aligned} n|P(z)| &= |zP'(z) + nP(z) - zP'(z)| \\ &\leq |P'(z)| + |nP(z) - zP'(z)| = |P'(z)| + |Q'(z)| \end{aligned}$$

for $|z| = 1$. (10) and (11) imply (9). In view of (11), it is obvious that if $|P(z_0)| = \max_{|z|=1} |P(z)|$ and $|z_0| = 1$, then $|P'(z_0)| + |Q'(z_0)| = n|P(z_0)|$. This proves Theorem 2.

If $P(z)$ is a self-inverse polynomial of degree n , i.e. $P(z) = z^n \overline{P(1/\bar{z})}$, from Theorem 2 we get an already known result:

THEOREM C. *If $P(z)$ is a self-inverse polynomial of degree n , then*

$$(12) \quad \max_{|z|=1} |P'(z)| = \frac{n}{2} \max_{|z|=1} |P(z)|.$$

Theorem C is proved by Govil [1], O'Hara and Rodrigues [6], and Saff and Sheil-Small [8] independent of each other. In this case, we further note that $|P'(z_0)| = (n/2)|P(z_0)|$, where $|z_0| = 1$ and $|P(z_0)| = \max_{|z|=1} |P(z)|$. This implies that if $|P(z)|$ attains its maximum on $|z| = 1$ at n points (n is the degree of $P(z)$) then $|P'(z)|$ must also attain its maximum at those n points. Then $P'(z)$ being a polynomial of degree $n - 1$ must reduce to $P'(z) = \alpha z^{n-1}$ where α is a constant. Consequently, we have

THEOREM 3. *If $P(z)$ is a self-inverse polynomial of degree n and $|P(z)|$ attains its maximum at n points on $|z| = 1$, then $P(z) = \alpha z^n + \bar{\alpha}$ where α is a constant.*

REMARK 3. Rubinstein [7] has recently proved Theorem 3 for the class of polynomials having all its zeros on $|z| = 1$.

Finally, we give an application of Theorem C to real trigonometric polynomials. If $t(\theta)$ is a real trigonometric polynomial of order n , then

$$(13) \quad e^{in\theta}t(\theta) = R(e^{i\theta})$$

where $R(z)$ is a self-inverse polynomial of degree $2n$. Differentiating (13) with respect to θ and applying Theorem C, we get

$$\max_{\theta} \sqrt{\{nt(\theta)\}^2 + \{t'(\theta)\}^2} = n \max_{\theta} |t(\theta)|,$$

an inequality essentially due to van der Corput and Schaake [10].

We thank Professor Q. I. Rahman for useful discussions.

REFERENCES

1. N. K. Govil, *On the derivative of a polynomial*, Proc. Amer. Math. Soc. **41** (1973), 543–546.
2. E. Hille, *Analytic function theory*, Vol. II, Blaisdell, Toronto, 1962.
3. P. D. Lax, *Proof of a conjecture of P. Erdős on the derivative of a polynomial*, Bull. Amer. Math. Soc. **50** (1944), 509–513.
4. G. R. MacLane, *Polynomials with zeros on a rectifiable Jordan curve*, Duke. Math. J. **16** (1949), 461–477.
5. M. A. Malik, *On the derivative of a polynomial*, J. London Math. Soc. **1** (1969), 57–60.
6. P. J. O'Hara and R. S. Rodrigues, *Some properties of a self-inverse polynomial*, Proc. Amer. Math. Soc. **44** (1974), 331–335.
7. Z. Rubinstein, Personal communication.
8. E. B. Saff and T. Sheil-Small, *Coefficient and integral mean estimates for algebraic and trigonometric polynomials with restricted zeros*, J. London Math. Soc. **9** (1974), 16–22.
9. P. Turan, *Ueber die ableitung von polynomen*, Compositio Math. **7** (1939), 89–95.
10. J. G. van der Corput and G. Schaake, *Ungleichungen für polynome und trigonometrische polynome*. Compositio Math. **2** (1935), 321–361.

DEPARTMENT OF MATHEMATICS, CONCORDIA UNIVERSITY, MONTREAL (H3G 1M8), QUEBEC, CANADA