A NONLOCALLY CONNECTED CONTINUUM X SUCH THAT $C(X)$ IS A RETRACT OF 2^X

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ABSTRACT. Let X be a metric continuum and let $2^X(C(X))$ denote the hyperspace of closed subsets (subcontinua) of X . An example is given of a nonlocally connected continuum X such that $C(X)$ is a retract of 2^X .

By a *continuum* we mean a compact connected metric space. If X is a continuum, then 2^X (respectively, $C(X)$) denotes the hyperspace of closed subsets (respectively, subcontinua) of X , each with the Hausdorff metric. In this note we will obtain a partial answer to a question raised by Sam B. Nadler, Jr. [8] by giving an example of a nonlocally connected continuum X such that $C(X)$ is a retract of 2^X .

In 1939 Wojdyslawski [10] proved that $C(X)$ is an absolute retract if and only if X is locally connected. Hence, $C(X)$ is a retract of 2^X when X is locally connected. In Theorem 3.6 of [8], Nadler shows that $C(X)$ is always a continuous image of 2^X . Following this result, in Question 3.7 of [8], Nadler asks, "When is $C(X)$ a retract of 2^X ?" Nadler discusses this question further in [6] and in Chapter 6 of [7]. In particular, in Theorem 6.11 of [7], Nadler proves a result which indicates that, when X is not locally connected, a retraction from 2^X onto $C(X)$, if one exists, must be rather complicated. In [2] the author determined the first partial answer to Nadler's question by giving an example of a nonlocally connected continuum X such that $C(X)$ is not a retract of 2^X . In [5] Lawson attempted to give some conditions which would imply the existence of a retraction from 2^X onto $C(X)$ for certain nonlocally connected continua. However, as noted in [2], Lawson's proof was not correct. We will show that for one of the examples in Lawson's paper, namely the cone over a convergent sequence, there does exist a retraction from 2^X onto $C(X)$. It is interesting to note that this example and the example in [2] are both smooth dendroids. Thus, even within this rather well-behaved class of continua, $C(X)$ may or may not be a retract of 2^X when X is not locally connected.

We observe in Corollary 1 that the example in this paper also has the property that $F_1(X)$, the space of singleton subsets of X , is a retract of 2^X . This is the first example of a nonlocally connected continuum for which it has been shown that $F_1(X)$ is a retract of 2^X . Hence, the example also gives a partial answer to Nadler's question in 6.2 of [7].

Received by the editors July 5, 1983 and, in revised form, September 8, 1983.

1980 *Mathematics Subject Classification*. Primary 54B20; Secondary 54C15.

Key words and phrases. Continuum, hyperspace, retract, smooth dendroid.

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0002-9939/84 \$1.00 + \$.25 per page

The symbol I will denote the closed unit interval of real numbers and the symbol $\mathbf{Z}^+$ will denote the set of positive integers. For the remainder of this paper, the symbol X will denote the specific continuum defined in the following paragraph. We will let d denote the usual Euclidean metric on X and ρ denote the Hausdorff metric on 2^X . An order arc β in $C(X)$ is an arc with the property that whenever $A, B \in \beta$, then $A \subset B$ or $B \subset A$.

We describe the planar continuum X in terms of polar coordinates. Let $T = \{0\} \cup \{1/n | n \in \mathbf{Z}^+\}$. For each $\theta \in T$, let $L_\theta = \{(r, \theta) | r \in I\}$. Let $X = \bigcup \{L_\theta | \theta \in T\}$, $p = (0, 0)$, $2_p^X = \{A \in 2^X | p \in A\}$, and $\mathcal{A} = 2_p^X \cup (\bigcup \{2^{L_\theta} | \theta \in T\})$. We note that $\mathcal{A}$ is closed and that $C(X) \subset \mathcal{A}$.

We now list all of the functions that we use to define a retraction R from 2^X onto $C(X)$. Define mappings $r_{\min}, r_{\max}: 2^X \rightarrow I$ by

$$r_{\min}(A) = \min\{r | (r, \theta) \in A\} \quad \text{and} \quad r_{\max}(A) = \max\{r | (r, \theta) \in A\}.$$

Define mappings $\theta_{\min}, \theta_{\max}: 2^X - 2_p^X \rightarrow I$ by

$$\theta_{\min}(A) = \min\{\theta | (r, \theta) \in A\} \quad \text{and} \quad \theta_{\max}(A) = \max\{\theta | (r, \theta) \in A\}.$$

Define the mapping $g: 2^X - 2_p^X \rightarrow C(X)$ by

$$g(A) = \{(r, \theta_{\min}(A)) | r_{\min}(A) \leq r \leq r_{\max}(A)\}.$$

Define the function $f: 2^X \rightarrow C(X)$ by

$$f(A) = \bigcap \{M \in C(X) | A \subset M\}.$$

Using Theorem 1 of [1] and Theorems 1, 2 and 3 of [4], it is easy to show that if $A \in 2^X$ and $A \not\subset L_0 - \{p\}$, then f is continuous at A . Define the mapping $\alpha: (2^X - \mathcal{A}) \times I \rightarrow C(X)$ by

$$\alpha(A, s) = \begin{cases} h_1(g(A), 3s) & \text{if } s \in [0, \frac{1}{3}], \\ f(\{p\} \cup g(A)) \cup h_2(A, 3s - 1) & \text{if } s \in [\frac{1}{3}, \frac{2}{3}], \\ f(A) \cup h_2(\{(r_{\max}(A), \theta_{\min}(A))\}, 3 - 3s) & \text{if } s \in [\frac{2}{3}, 1], \end{cases}$$

where h_1 and h_2 are as defined on p. 36 of [3]. If $A \in 2^X - \mathcal{A}$, then $\{\alpha(A, s) | s \in [0, \frac{1}{3}]\}$ is the unique order arc in $C(X)$ from $g(A)$ to $f(\{p\} \cup g(A))$, $\{\alpha(A, s) | s \in [\frac{1}{3}, \frac{2}{3}]\}$ is the order arc in $C(X)$ which grows uniformly from $f(\{p\} \cup g(A))$ to $f(\{p\} \cup g(A)) \cup f(A)$, and $\{\alpha(A, s) | s \in [\frac{2}{3}, 1]\}$ is the unique order arc (possibly degenerate) in $C(X)$ from $f(A)$ to $f(\{p\} \cup g(A)) \cup f(A)$. Thus, for each $A \in 2^X - \mathcal{A}$, $\{\alpha(A, s) | s \in I\}$ is an arc in $C(X)$ from $g(A)$ to $f(A)$. Define the function $R: 2^X \rightarrow C(X)$ by

$$(1) \quad R(A) = \begin{cases} f(A) & \text{if } A \in \mathcal{A}, \\ \alpha(A, \min\{\theta_{\max}(A)/r_{\min}(A), 1\}) & \text{if } A \in 2^X - \mathcal{A}. \end{cases}$$

THEOREM 1. *Let R be as defined in (1). Then R is a retraction from 2^X onto $C(X)$.*

PROOF. It is clear that R is continuous on the open set $2^X - \mathcal{A}$. Let $A \in \mathcal{A}$. If $A \in 2_p^X$, then f is continuous at A . If $A \in \mathcal{A} - 2_p^X$, then $f(A) = g(A)$. Since g is continuous at A , it follows that the restriction of f to the closed set $\mathcal{A}$ is continuous at A . Hence, the restriction of R to the closed set $\mathcal{A}$ is continuous. Thus, to complete

the proof that R is continuous, it will suffice to show that if $A \in \mathcal{A}$ and $\{A_n\}_{n=1}^\infty$ is a sequence in $2^X - \mathcal{A}$ such that $A_n \rightarrow A$, then $R(A_n) \rightarrow R(A)$. Let $\varepsilon > 0$. For $A \in \mathcal{A}$, we will consider three distinct cases.

Case I. Suppose that $p \in A$ and $A \not\subset L_0$. Since f is continuous at A , there exists $\delta > 0$ such that if $B \in 2^X$ and $\rho(A, B) < \delta$, then $\rho(f(A), f(B)) < \varepsilon$. Since $A_n \rightarrow A$, there exists $n_1 \in \mathbb{Z}^+$ such that $n \geq n_1$ implies $\rho(A_n, A) < \delta/2$. For some $\theta > 0$, $A \cap (L_\theta - \{p\}) \neq \emptyset$. It follows that there exists $n_2 \in \mathbb{Z}^+$ such that $n \geq n_2$ implies $\theta_{\max}(A_n) > \theta/2$. Thus, since $r_{\min}(A_n) \rightarrow r_{\min}(A) = 0$, there exists $n_3 \in \mathbb{Z}^+$ such that $n \geq n_3$ implies $\theta_{\max}(A_n)/r_{\min}(A_n) > 1$. Let $n \in \mathbb{Z}^+$ such that $n \geq \max\{n_1, n_3\}$. Then

$$R(A_n) = \alpha(A_n, 1) = f(A_n) \quad \text{and} \quad \rho(R(A_n), R(A)) = \rho(f(A_n), f(A)) < \varepsilon.$$

Case II. Suppose that $p \in A$ and $A \subset L_0$. Let δ and n_1 be as in Case I. Since $r_{\min}(A_n) \rightarrow r_{\min}(A) = 0$, there exists $n_4 \in \mathbb{Z}^+$ such that $n \geq n_4$ implies $r_{\min}(A_n) < \delta/2$. Let $n \in \mathbb{Z}^+$ such that $n \geq \max\{n_1, n_4\}$. If $\theta_{\max}(A_n) \geq r_{\min}(A_n)$, then

$$R(A_n) = \alpha(A_n, 1) = f(A_n) \quad \text{and} \quad \rho(R(A_n), R(A)) = \rho(f(A_n), f(A)) < \varepsilon.$$

Now suppose that $\theta_{\max}(A_n) \leq r_{\min}(A_n) < \delta/2$. Let $\pi(A_n) = \{(r, \theta_{\min}(A_n)) | (r, \theta) \in A_n\}$. Since $\theta_{\min}(A_n) \leq \theta_{\max}(A_n) < \delta/2$, $\rho(\pi(A_n), A_n) < \delta/2$. Thus,

$$\rho(\pi(A_n), A) \leq \rho(\pi(A_n), A_n) + \rho(A_n, A) < \delta/2 + \delta/2 = \delta.$$

Hence, $\rho(f(\pi(A_n)), f(A)) = \rho(g(A_n), f(A)) < \varepsilon$. Since $\rho(\pi(A_n), A) < \delta$ and since $p \in A$, $\rho(\{p\} \cup \pi(A_n), A) < \delta$. Hence,

$$\rho(f(\{p\} \cup \pi(A_n)), f(A)) = \rho(f(\{p\} \cup g(A_n)), f(A)) < \varepsilon.$$

Since $\rho(f(\{p\} \cup g(A_n)), f(A)) < \varepsilon$ and $\rho(f(A_n), f(A)) < \varepsilon$,

$$\rho(f(\{p\} \cup g(A_n)) \cup f(A_n), f(A)) < \varepsilon.$$

Let $s \in I$. Then $g(A_n) \subset \alpha(A_n, s) \subset f(\{p\} \cup g(A_n))$ or $f(\{p\} \cup g(A_n)) \subset \alpha(A_n, s) \subset f(\{p\} \cup g(A_n)) \cup f(A_n)$ or $f(A_n) \subset \alpha(A_n, s) \subset f(\{p\} \cup g(A_n)) \cup f(A_n)$. It follows that for each $s \in I$, $\rho(\alpha(A_n, s), f(A)) < \varepsilon$. Since $R(A_n) \in \{\alpha(A_n, s) | s \in I\}$, we conclude that

$$\rho(R(A_n), R(A)) = \rho(R(A_n), f(A)) < \varepsilon.$$

Case III. Suppose that $p \notin A$. Since $\{A_n\}_{n=1}^\infty \subset 2^X - \mathcal{A}$ and since $A_n \rightarrow A$, it follows that $A \subset L_0 - \{p\}$. Then $R(A) = f(A) = g(A)$. Since h_1 is uniformly continuous, there exists $\delta_1 > 0$ such that if $B_1, B_2 \in C(X)$ and $s_1, s_2 \in I$ such that $\rho(B_1, B_2) < \delta_1$ and $|s_1 - s_2| < \delta_1$, then $\rho(h_1(B_1, s_1), h_1(B_2, s_2)) < \varepsilon$. Since g is continuous at A , there exists $n_5 \in \mathbb{Z}^+$ such that $n \geq n_5$ implies $\rho(g(A_n), g(A)) < \delta_1$. Since $\theta_{\max}(A_n) \rightarrow \theta_{\max}(A) = 0$ and $r_{\min}(A_n) \rightarrow r_{\min}(A) > 0$, there exists $n_6 \in \mathbb{Z}^+$ such that $n \geq n_6$ implies $\theta_{\max}(A_n)/r_{\min}(A_n) < \min\{\delta_1/3, 1/3\}$. Let $n \in \mathbb{Z}^+$ such that $n \geq \max\{n_5, n_6\}$. Then

$$\begin{aligned} R(A_n) &= \alpha(A_n, \theta_{\max}(A_n)/r_{\min}(A_n)) \\ &= h_1(g(A_n), 3(\theta_{\max}(A_n)/r_{\min}(A_n))). \end{aligned}$$

Since $\rho(g(A_n), g(A)) < \delta_1$ and

$$|3(\theta_{\max}(A_n)/r_{\min}(A_n)) - 0| < 3(\delta_1/3) = \delta_1$$

and since $h_1(g(A), 0) = g(A)$,

$$\rho(h_1(g(A_n), 3(\theta_{\max}(A_n)/r_{\min}(A_n))), h_1(g(A), 0)) = \rho(R(A_n), R(A)) < \varepsilon.$$

The proof that R is continuous is now complete. Since $C(X) \subset \mathcal{A}$, $R(A) = f(A) = A$ for each $A \in C(X)$. Hence, R is a retraction from 2^X onto $C(X)$.

COROLLARY 1. *Let $F_1(X) = \{\{x\} | x \in X\}$. Then $F_1(X)$ is a retract of 2^X .*

PROOF. Since X is a smooth dendroid, it follows from Theorem 1 of [9] that $F_1(X)$ is a retract of $C(X)$. By Theorem 1, $C(X)$ is a retract of 2^X . Hence, $F_1(X)$ is a retract of 2^X .

REFERENCES

1. J. T. Goodykoontz, Jr., *Connectedness im kleinen and local connectedness in 2^X and $C(X)$* , Pacific J. Math. **53** (1974), 387–397.
2. ———, *$C(X)$ is not necessarily a retract of 2^X* , Proc. Amer. Math. Soc. **67** (1977), 177–178.
3. ———, *Hyperspaces of arc-smooth continua*, Houston J. Math. **7** (1981), 33–41.
4. ———, *Some functions on hyperspaces of hereditarily unicoherent continua*, Fund. Math. **95** (1977), 1–10.
5. J. D. Lawson, *Applications of topological algebra to hyperspace problems*, Lecture Notes in Pure and Appl. Math., No. 24, Marcel Dekker, New York, 1976, pp. 201–206.
6. S. B. Nadler, Jr., *A characterization of locally connected continua by hyperspace retractions*, Proc. Amer. Math. Soc. **67** (1977), 167–176.
7. ———, *Hyperspaces of sets*, Dekker, New York, 1978.
8. ———, *Some problems concerning hyperspaces*, Lecture Notes in Math., Vol. 375, Springer-Verlag, New York, 1974, pp. 190–197.
9. S. B. Nadler, Jr., and L. E. Ward, Jr., *Concerning continuous selections*, Proc. Amer. Math. Soc. **25** (1970), 369–374.
10. M. Wojdyslawski, *Retracts absolus et hyperspaces des continus*, Fund. Math. **32** (1939), 184–192.

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