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RESTRICTION OF SPHERICAL REPRESENTATIONS OF $\mathrm{PGL}_2(\mathbf{Q}_p)$ TO A DISCRETE SUBGROUP

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ABSTRACT. We study the action on the homogeneous tree associated with $\mathrm{PGL}_2(\mathbf{Q}_p)$, of a suitably chosen discrete subgroup which is isomorphic to $\mathbf{Z}_2 * \cdots * \mathbf{Z}_2$ and is cocompact. We prove that the spherical representations of $\mathrm{PGL}_2(\mathbf{Q}_p)$ remain irreducible when restricted to this subgroup.

Introduction. P. Cartier's combinatorial approach to the theory of representations of $\mathrm{PGL}_2(\mathbf{Q}_p)$ (the two by two projective linear group over a p-adic field $\mathbf{Q}_p$) is based on the realization of the group as a group of isometries of a homogeneous tree T of order $p+1$ [5]. One defines a boundary Ω of T and a probability measure on Ω by means of the simple random walk. The action of $\mathrm{PGL}_2(\mathbf{Q}_p)$ on T induces an action on Ω, and the measure ν is quasi-invariant with respect to the action of $\mathrm{PGL}_2(\mathbf{Q}_p)$. That is, letting $\nu_g(E) = \nu(gE)$, the measure ν_g is absolutely continuous with respect to ν, and one may define a Poisson kernel as the Radon-Nikodým derivative $P(g, \omega) = d\nu_g(\omega)/d\nu$, for $\omega \in \Omega$. Finally, if $z \in \mathbf{C}$, one defines the *spherical representations* of $\mathrm{PGL}_2(\mathbf{Q}_p)$, on the space of simple functions on Ω, by the formula

$$(\pi_z(g)\xi)(\omega) = P^z(g, \omega)\xi(g^{-1}\omega).$$

Spherical functions are then obtained integrating $P^z(g, \omega)$ over the boundary.

The same approach was more recently applied to certain discrete groups of isometries of T. In particular (for a homogeneous tree of order $p+1$), to the free group $\mathbf{F}((p+1)/2)$ on $(p+1)/2$ generators, and to the free product $\mathbf{Z}_2 * \cdots * \mathbf{Z}_2$ ($p+1$ times) of the two element group $\mathbf{Z}_2$. The graph naturally associated with such a group can be put in one-to-one correspondence with T, in such a way that the vertices of T correspond to the elements of the group. The action on T resulting from left multiplication turns out to be isometric, simply transitive and without fixed points. Spherical representation and spherical functions are then defined as above and one proves that the spherical unitary representations are *irreducible* [5, 7].

On the other hand $\mathrm{PGL}_2(\mathbf{Q}_p)$ contains discrete subgroups isomorphic to free groups or free products. It is natural therefore to ask whether a direct connection exists between the spherical representations of $\mathrm{PGL}_2(\mathbf{Q}_p)$ and those of its discrete subgroups.

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In this note we exhibit a discrete subgroup Γ of $\mathrm{PGL}_2(\mathbf{Q}_p)$, with compact quotient, which is isomorphic to $\mathbf{Z}_2 * \cdots * \mathbf{Z}_2$ ($p+1$ times) and which acts in exactly the same way on its associated graph and on the tree of $\mathrm{PGL}_2(\mathbf{Q}_p)$. The latter group splits in the product of Γ and the compact stabilizer of a vertex of the tree.

Because spherical representations both in $\mathrm{PGL}_2(\mathbf{Q}_p)$ and Γ are defined solely by means of the group action on T, this implies that the spherical representations of $\mathrm{PGL}_2(\mathbf{Q}_p)$ restrict to spherical representations of Γ. In particular, irreducibility is preserved under restriction to Γ for the unitary spherical representations of $\mathrm{PGL}_2(\mathbf{Q}_p)$.

This result suggests a more general question, which is not explored in this note. Let G be a reductive group and Γ a discrete subgroup such that G/Γ has finite volume. Let π be an irreducible unitary representation of G: when is the restriction of π to Γ irreducible?

This question is unresolved even for the classical case of $\mathrm{SL}(2,\mathbf{R})$ with $\Gamma = \mathrm{SL}(2,\mathbf{Z})$. One does know that, if π is an element of the discrete series of $\mathrm{SL}(2,\mathbf{R})$, then $\pi|_{\mathrm{SL}(2,\mathbf{Z})}$ is *not* irreducible (no infinite discrete group can have an irreducible subrepresentation of its regular representation). On the other hand, if π is a spherical representation of $\mathrm{SL}(2,\mathbf{R})$, it is unknown, so far as we know, whether the restriction of π to $\mathrm{SL}(2,\mathbf{Z})$ is irreducible. In fact, Helgason has shown that such a restriction is *not* algebraically irreducible (see [**8**, §5.5]), although it still, of course, may be topologically irreducible (the usual sense of irreducibility). In our first corollary, we show that $\pi|_\Gamma$ is irreducible when π is a unitary spherical representation of $\mathrm{PGL}_2(\mathbf{Q}_p)$ and Γ is a particular lattice.

Spherical representations of $\mathrm{PGL}_2(\mathbf{Q}_p)$ are described in [**11**] without reference to trees. J. P. Serre's original monograph [**12**] is the best reference for the general theory of groups acting on trees. The harmonic analysis on trees with special reference to p-adic groups was developed by P. Cartier in [**3-5**]. Spherical functions and spherical representations of free groups are treated in [**7, 8**] (see also [**6, 10**]). This theory was extended to $\mathbf{Z}_2 * \cdots * \mathbf{Z}_2$ in [**2**]. Other related references may be found in [**8**].

The tree associated with $\mathrm{PGL}_2(\mathbf{Q}_p)$. Let K be the subgroup of $\mathrm{SL}_2(\mathbf{Q}_p)$ consisting of matrices with entries in the ring $\mathcal{O}_p$ of p-adic integers. Let $w = \left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$ and $K' = wKw^{-1} \subset \mathrm{SL}_2(\mathbf{Q}_p)$. We introduce a tree of *cosets* as follows. The vertices are the cosets $\{gK\colon g \in \mathrm{SL}_2(\mathbf{Q}_p)\} \cup \{gK'\colon g \in \mathrm{SL}_2(\mathbf{Q}_p)\}$. Two vertices belong to the same edge if their intersection is nonempty [**1**, p. 262]. On this tree $\mathrm{SL}_2(\mathbf{Q}_p)$ acts by left multiplication. The stabilizer of a given vertex under this action is a subgroup of the type gKg^{-1} or $gK'g^{-1}$, with $g \in \mathrm{SL}_2(\mathbf{Q}_p)$. Thus

$$\left\{gKg^{-1}\colon g \in \mathrm{SL}_2(\mathbf{Q}_p)\right\} \cup \left\{gK'g^{-1}\colon g \in \mathrm{SL}_2(\mathbf{Q}_p)\right\} = \left\{gKg^{-1}\colon g \in \mathrm{GL}_2(\mathbf{Q}_p)\right\}$$

can be identified with the set of vertices of a tree T isomorphic with the tree of cosets. The group $\mathrm{GL}_2(\mathbf{Q}_p)$ acts on T by conjugation: the scalar matrices act trivially, and therefore $\mathrm{PGL}_2(\mathbf{Q}_p)$ is isomorphic to a group of isometries of T. Denote by $|\ |_p$ the p-adic norm, and let $K_0 = K \cap K' = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K\colon |c|_p < 1\}$. The index of K_0 in K, and in wKw^{-1}, is $p+1$. Let $a_0 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $a_j = \left(\begin{smallmatrix} 1 & 0 \\ j & 1 \end{smallmatrix}\right)$, for

$j = 1,\dots,p$. Then a_jK_0, $j = 0,\dots,p$, are distinct cosets of K_0 in K, and $K = \bigcup_{j=0}^{p} a_jK_0$ ($a_p \in K_0$ and $a_pK_0 = K_0$). The nearest neighbors of K are the $p+1$ *distinct* vertices $a_jwKw^{-1}a_j^{-1}$, $j = 0,\dots,p$, because $K \cap a_jwKw^{-1} = a_jK_0$.

The tree associated with $\mathbf{Z}_2 * \cdots * \mathbf{Z}_2$. We consider the free product $\Gamma = \mathbf{Z}_2 * \cdots * \mathbf{Z}_2$ of $p+1$ copies of the two element group $\mathbf{Z}_2$. We define a tree as follows: let $\{x_0, x_1,\dots,x_p\}$ be a given set of generators of Γ satisfying $x_j^2 = e$. We let the vertices of T be the elements of Γ. Two elements $x, y \in \Gamma$ belong to the same edge if $x = yx_j$, for some $j = 0,\dots,p$. The nearest neighbors of the identity e are the generators $\{x_0,\dots,x_p\}$. Every element of Γ can be written uniquely as a reduced word in the generators x_j, $0 \leqslant j \leqslant p$, therefore T is a tree [**2, 12**].

The group Γ acts isometrically on T by left multiplication.

THEOREM. *There exists a closed discrete subgroup Γ of* $\mathrm{PGL}_2(\mathbf{Q}_p)$ *isomorphic to* $\mathbf{Z}_2 * \cdots * \mathbf{Z}_2$ *($p+1$ times), and such that the action of Γ on the tree of* $\mathrm{PGL}_2(\mathbf{Q}_p)$ *coincides with the action of $\mathbf{Z}_2 * \cdots * \mathbf{Z}_2$ on its associated tree, as described above.*

PROOF. Let Γ be the subgroup of $\mathrm{PGL}_2(\mathbf{Q}_p)$ generated by $x_j = a_jwa_j^{-1}$, $j = 0,\dots,p$, so that $x_jKx_j^{-1} = a_jwKw^{-1}a_j^{-1}$ are the nearest neighbors of K in the tree T associated with $\mathrm{PGL}_2(\mathbf{Q}_p)$. Observe that $x_pKx_p^{-1} = wKw^{-1}$, since $w^{-1}a_pw = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \in K$. We show that Γ acts transitively on T, in other words that Γ carries, under conjugation, K into any other vertex gKg^{-1} of T. Let gKg^{-1} be a vertex of T. If gKg^{-1} has distance one from K then

$$gKg^{-1} = a_jwKw^{-1}a_j^{-1} = a_jwa_j^{-1}Ka_jw^{-1}a_j^{-1} = x_jKx_j^{-1}$$

for some $j = 0,\dots,p$. By induction, suppose that $gKg^{-1} \in \{xKx^{-1}\colon x \in \Gamma\}$ for every $g \in \mathrm{PGL}_2(\mathbf{Q}_p)$, such that gKg^{-1} has distance $n-1$ from K. Now let gKg^{-1} have distance n from K. Then there exists an element of the type $x_jKx_j^{-1}$ which has distance $n-1$ from gKg^{-1}. It follows that $x_j^{-1}gKg^{-1}x_j$ has distance $n-1$ from K, and, by induction hypothesis, $x_j^{-1}gKg^{-1}x_j = xKx^{-1}$ for some $x \in \Gamma$. Therefore $gKg^{-1} = x_jxKx^{-1}x_j^{-1}$, and $x_jx \in \Gamma$. We show now that $xKx^{-1} = K$, for $x \in \Gamma$, implies that x is the identity. Let $x \in \Gamma$, $x \neq e$. Write $x = x_{j_1} \cdots x_{j_s}$, and $x^{-1} = x_{j_s} \cdots x_{j_1}$ with $j_i \neq j_{i+1}$; this is possible since $x_j = x_j^{-1}$. Suppose that $xKx^{-1} = x_{j_1} \cdots x_{j_s}Kx_{j_s} \cdots x_{j_1} = K$, and let $y_t = x_{j_1} \cdots x_{j_t}$. Then the sequence $K, y_1Ky_1^{-1},\dots,y_sKy_s^{-1}$ is a loop in T. The loop must be trivial because T is a tree, and therefore, for some i, $y_{i+1}Ky_{i+1}^{-1} = y_{i-1}Ky_{i-1}^{-1}$. Thus

$$y_ix_{j_{i+1}}Kx_{j_{i+1}}y_i^{-1} = y_{i-1}Ky_{i-1}^{-1} = y_ix_{j_i}Kx_{j_i}y_i^{-1}.$$

Therefore $x_{j_{i+1}}Kx_{j_{i+1}} = x_{j_i}Kx_{j_i}$, which implies $x_{j_{i+1}} = x_{j_i}$. This contradicts the hypothesis that $j_i \neq j_{i+1}$. We have thus proved that the map $x \to xKx^{-1}$ is a bijection of Γ onto the vertices of T, and also that Γ is isomorphic to the group of reduced words in the generators $x_0,\dots,x_p$, that is Γ is isomorphic to the free product of $p+1$ copies of $\mathbf{Z}_2$. Under the correspondence $x \to xKx^{-1}$, left multiplication on Γ, $x \to yx$, corresponds to the action $y(xKx^{-1})y^{-1}$ on the tree. This shows that the action of Γ on its associated graph is the same as the action of Γ on T. Finally, let

$G_0 = \{g \in \mathrm{PGL}_2(\mathbf{Q}_p)\colon gKg^{-1} = K\}$. Then G_0 is open and compact and the proof above shows that $G_0 \cap \Gamma = \{e\}$. This implies that Γ is discrete and that $\mathrm{PGL}_2(\mathbf{Q}_p) = G_0\Gamma$.

COROLLARY. *Let π be a unitary spherical representation of* $\mathrm{PGL}_2(\mathbf{Q}_p)$. *Then the restriction of π to the discrete subgroup Γ is irreducible.*

PROOF. The corollary follows from the fact that unitary spherical representations of $\mathbf{Z}_2 * \cdots * \mathbf{Z}_2$ are irreducible [**2**].

The next result could be obtained directly, adapting the techniques of [**9**]. It is also a consequence of the corresponding result for $\mathrm{PGL}_2(\mathbf{Q}_p)$, which was proved in [**11**].

COROLLARY. *Let $0 \leqslant \operatorname{Re} z \leqslant 1$ and let π_z be a spherical representation of $\mathbf{Z}_2 * \cdots * \mathbf{Z}_2$ ($p + 1$ times), as defined in* [**2**]. *Then π_z is uniformly bounded.*

REFERENCES

1. R. Alperin, *Locally compact groups acting on trees and property T*, Monatsh. Math. **93** (1982), 261–265.

2. W. Betori and M. Pagliacci, *Harmonic analysis for groups acting on trees*, Boll. Un. Mat. Ital. (to appear).

3. P. Cartier, *Fonctions harmoniques sur un arbre*, Symposia Mathematica **9** (1972), 203–270.

4. ______, *Harmonic analysis on trees*, Proc. Sympos. Pure Math., vol. 26, Amer. Math. Soc., Providence, R.I., 1972, pp. 419–424.

5. ______, *Géomètrie et analyse sur les arbres*, Lecture Notes in Math., vol. 317, Springer-Verlag, Berlin, 1973.

6. J. M. Cohen, *Operator norms on free groups*, Boll. Un. Mat. Ital. B **1** (1982), 1055–1065.

7. A. Figà-Talamanca and M. A. Picardello, *Spherical functions and harmonic analysis on free groups*, J. Funct. Anal. **47** (1982), 281–304.

8. ______, *Harmonic analysis on free groups*, Lecture Notes in Pure and Applied Math., vol. 87, Marcel Dekker, New York, 1983.

9. A. M. Mantero and A. Zappa, *The Poisson transform and representations of a free group*, J. Funct. Anal. **51** (1983), 372–399.

10. T. Pytlik, *Radial functions on free groups and a decomposition of the regular representation into irreducible components*, J. Reine Angew. Math. **326** (1981), 124–135.

11. P. J. Sally, *Unitary and uniformly bounded representations of the two by two unimodular group over local fields*, Amer. J. Math. **90** (1968), 406–443.

12. J. P. Serre, *Arbres, amalgames*, SL_2, Astérisque **46** (1977).

13. A. J. Silberger, PGL_2 *over the p-adics: its representations, spherical functions and Fourier analysis*, Lecture Notes in Math., vol. 166, Springer-Verlag, Berlin, 1970.

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