ELEMENTARY PROOF OF A FORMULA OF RAMANUJAN

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ABSTRACT. In this paper we use only elementary calculus to prove Ramanujan's integral formula

$$\int_0^\infty x^{m-1} \frac{(1+abx)(1+ab^2x) \cdots (1+ab^nx) \cdots}{(1+x)(1+bx)(1+b^2x) \cdots (1+b^nx) \cdots} dx \\ = \frac{\pi}{\sin(m\pi)} \prod_{k=1}^\infty \frac{(1-b^{k-m})(1-ab^k)}{(1-b^k)(1-ab^{k-m})},$$

where m , a and b are positive with $b < 1$ and $a < b^{m-1}$.

1. Introduction. In Ramanujan's collected papers [5, p. 57], the following elegant formula is found:

$$(1) \quad \int_0^\infty x^{m-1} \frac{(1+abx)(1+ab^2x) \cdots (1+ab^nx) \cdots}{(1+x)(1+bx)(1+b^2x) \cdots (1+b^nx) \cdots} dx \\ = \frac{\pi}{\sin(m\pi)} \prod_{k=1}^\infty \frac{(1-b^{k-m})(1-ab^k)}{(1-b^k)(1-ab^{k-m})}$$

where m , a and b are positive with $b < 1$ and $0 < a < b^{m-1}$.

Ramanujan did not prove his formula. G. H. Hardy [3], using complex variable theory, gave the first proof of it in 1915. And more recently, another evaluation of (1) has been given by R. A. Askey [1]. The purpose of this paper is to give another proof of Ramanujan's formula, but one which does not use any mathematics higher than calculus.

In our proof of (1) we shall assume that m is not an integer and a is not an integral power of b . The formulas of these exceptional cases may be found in Ramanujan's paper and are readily deduced from (1).

2. Proof that Ramanujan's integral converges. If we write

$$R(u) = \int_0^u x^{m-1} \frac{B(x)}{1+x} dx, \quad \text{where } B(x) = \prod_{k=1}^\infty \frac{1+ab^kx}{1+b^kx},$$

then, to prove convergence, we need only show that there is a constant A , such that $R(u) < A$ for all values of u greater than 0 [4, p. 358].

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Without loss of generality, we assume that $u > 1$. Let

$$(2) \quad T_m(u) = \int_1^u x^{m-1} \frac{B(x)}{1+x}.$$

Then, by putting $x^{m-1} = x^{m-2}(1+x-1)$, we obtain

$$(3) \quad T_m(u) = \int_1^u x^{m-2} B(x) dx - T_{m-1}(u).$$

Now, changing bx to x , we have

$$\begin{aligned} \int_1^u x^{m-2} B(x) dx &= b^{1-m} \int_b^{bu} x^{m-2} \frac{1+ax}{1+x} B(x) dx \\ &= b^{1-m} \int_b^{bu} x^{m-2} \frac{B(x)}{1+x} dx + ab^{1-m} \int_b^{bu} x^{m-1} \frac{B(x)}{1+x} dx. \end{aligned}$$

Let $p = [m]$, in which $[m]$ means the greatest integer $\leq m$. Then since $(1+ab^i x)/(1+b^{p-1+i} x) < 1$, for all positive integers i , it follows that

$$(4) \quad \frac{B(x)}{1+x} < \frac{1}{1+x} \prod_{k=1}^{p-1} \frac{1}{1+b^k x} < x^{-p} b^{-p(p-1)/2},$$

and hence

$$\int_b^{bu} x^{m-2} \frac{B(x)}{1+x} dx < \frac{b^{m-1-p(p+1)/2}}{p+1-m}.$$

And since

$$\int_b^{bu} x^{m-1} \frac{B(x)}{1+x} dx < R(u),$$

it follows that

$$(5) \quad \int_1^u x^{m-2} B(x) dx < \frac{b^{-p(p+1)/2}}{p+1-m} + ab^{1-m} R(u).$$

Noting that $T_{m-1}(u) > 0$, we have, by (3) and (5),

$$(6) \quad T_m(u) < \int_1^u x^{m-2} B(x) dx < \frac{b^{-p(p+1)/2}}{p+1-m} + ab^{1-m} R(u).$$

Now, from the definition of $R(u)$, $R(u) = R(1) + T_m(u)$. Thus,

$$R(u) < R(1) + \frac{b^{-p(p+1)/2}}{p+1-m} + ab^{1-m} R(u),$$

or $R(u) < A$, where

$$A = \frac{1}{1-ab^{1-m}} \left(R(1) + \frac{b^{-p(p+1)/2}}{p+1-m} \right) \quad \text{and} \quad u > 1.$$

Hence, since $R(u) \leq R(1)$ when $0 < u \leq 1$, we have $R(u) < A$ for all $u > 0$.

3. Evaluation of I_r . We require in our demonstration of (1) the value of

$$(7) \quad I_r = \int_0^\infty x^{m-1} \frac{D(x, r)}{1+x} dx, \quad \text{where} \quad D(x, r) = \prod_{k=1}^r \frac{1}{1+b^k x},$$

and $r > p = [m]$. By changing the independent variable x to bx , we have

$$I_r = b^m \int_0^\infty x^{m-1} (1+x) \frac{D(x, r+1)}{1+x} dx.$$

Next, using the identity $1+x = 1 + ((1+b^{r+1}x) - 1)/b^{r+1}$, we obtain

$$I_r = b^{m-r-1} I_r + b^{m-r-1} (b^{r+1} - 1) I_{r+1}.$$

Thus, $I_{r+1} = (1 - b^{r+1-m}) I_r / (1 - b^{r+1})$.

Iteration of our last equation gives us

$$I_r = I_p \prod_{k=1+p}^r \frac{1 - b^{k-m}}{1 - b^k}.$$

But, our reduction formula cannot be used when $r < p$, since the integrals I_{p-1} , I_{p-2} , etc. are divergent. We must now evaluate I_p .

Let $L_r = \int_0^\infty x^{\alpha-1} F_r(x) dx$, where $\alpha = m - p$, $0 \leq r \leq p$, and

$$F_r(x) = x^r \prod_{k=0}^r \frac{1}{1 + b^k} x.$$

Then observe that $L_p = I_p$.

Now, replace x by bx to obtain

$$L_p = b^{p+\alpha} \int_0^\infty F_p(x) \frac{1+x}{1+b^{p+1}x} x^{\alpha-1} dx.$$

Then changing p to $p-j$, where j is a positive integer $< p$, we have

$$L_{p-j} = b^{m-j} \int_0^\infty F_{p-j}(x) \frac{1+x}{1+b^{p-j+1}x} x^{\alpha-1} dx.$$

Thus,

$$(8) \quad b^{j-m} L_{p-j} - L_{p-j+1} = \int_0^\infty \frac{F_{p-j}(x)}{1+b^{p-j+1}x} x^{\alpha-1} dx.$$

But, L_p can also be rewritten as

$$L_p = \int_0^\infty F_{p-1}(x) \frac{x}{1+b^p x} x^{\alpha-1} dx,$$

and therefore, replacing p by $p-j+1$, multiplying both sides of the equation by b^{p+1-j} , and subtracting the result from L_{p-j} , we have

$$(9) \quad L_{p-j} - b^{p-j+1} L_{p-j+1} = \int_0^\infty \frac{F_{p-j}(x)}{1+b^{p-j+1}x} x^{\alpha-1} dx.$$

Next, eliminating the common integral from (8) and (9) yields the recurrence relation

$$L_{p-j+1} = -\frac{1-b^{j-m}}{1-b^{p-j+1}} L_{p-j}$$

from which we get

$$L_p = (-1)^p \prod_{k=1}^p \frac{1-b^{k-m}}{1-b^k} \int_0^\infty \frac{x^{\alpha-1}}{1+x} dx = \frac{\pi}{\sin m\pi} \prod_{k=1}^p \frac{1-b^{k-m}}{1-b^k},$$

since [2, p. 61]

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x} dx = \frac{\pi}{\sin \alpha\pi} = (-1)^p \frac{\pi}{\sin m\pi}.$$

Hence, collecting our results, we finally have

$$(10) \quad I_r = \frac{\pi}{\sin m\pi} \prod_{k=1}^r \frac{1 - b^{k-m}}{1 - b^k},$$

for all integers $r > 0$.

4. Proof of Ramanujan's formula. Let $I(a, b)$ denote the integral in (1). Then, replacing $1 + abx$ by $1 - ab + ab(1 + x)$ and using the fact that $I(a, b)$ is convergent, we have

$$I(a, b) = (1 - ab) \int_0^\infty \frac{x^{m-1}}{1+x} \prod_{k=1}^\infty \frac{1 + ab^{k+1}x}{1 + b^kx} + ab \int_0^\infty x^{m-1} \prod_{k=1}^\infty \frac{1 + ab^{k+1}x}{1 + b^kx} dx,$$

or, by changing bx to x in the second integral

$$I(a, b) = (1 - ab)I(ab, b) + ab^{1-m}I(a, b),$$

and hence

$$I(ab, b) = \frac{1 - ab^{1-m}}{1 - ab} I(a, b).$$

Thus,

$$(11) \quad I(ab^r, b) = I(a, b) \prod_{k=1}^r \frac{1 - ab^{k-m}}{1 - ab^k},$$

and on rewriting (11), we have

$$\begin{aligned} I(a, b) &= \frac{\pi}{\sin m\pi} \left\{ \prod_{k=1}^r \frac{(1 - b^{k-m})(1 - ab^k)}{(1 - b^k)(1 - ab^{k-m})} \right\} \\ &\quad \times \frac{I(ab^r, b)}{(\pi/\sin m\pi) \prod_{k=1}^r ((1 - b^{k-m})/(1 - b^k))}. \end{aligned}$$

Therefore, if we can show

$$(12) \quad \lim_{r \rightarrow \infty} \frac{I(ab^r, b)}{(\pi/\sin m\pi) \prod_{k=1}^r ((1 - b^{k-m})/(1 - b^k))} = 1,$$

then the proof of Ramanujan's formula will be complete.

In order to show that the limit in (12) is 1, we need the following two inequalities:

$$\begin{aligned} &\frac{1}{(1+x)(1+bx)(1+b^2x) \cdots (1+b^{r-2}x)} \\ &> \frac{(1+ab^r x)(1+ab^{r+1}x) \cdots (1+ab^{r+n+1}x)}{(1+x)(1+bx)(1+b^2x) \cdots (1+b^{r+n}x)} \end{aligned}$$

and

$$\frac{(1 + ab^r x)(1 + ab^{r+1} x) \cdots (1 + ab^{r+n} x)}{(1 + x)(1 + bx)(1 + b^2 x) \cdots (1 + b^{r+n+s} x)} \\ > \frac{1}{(1 + x)(1 + bx)(1 + b^2 x) \cdots (1 + b^{r+s-1} x)},$$

where n , r and s are positive integers, with $r > p = [m]$ and $0 < b^s < a < b^{-1}$, and with n independent of both r and s .

These inequalities can be easily verified; to establish the first inequality, replace a with b^{-1} and use the fact that $a < b^{-1}$; and to establish the second, replace a with b^s and recall $b^s < a$.

Now, multiply the above inequalities by x^{m-1} and let $n \rightarrow \infty$; then integrating yields the inequality $I_{r-2} \geq I(ab^{r-1}, b) \geq I_{r+s-1}$. From equation (10), we have

$$\frac{\pi}{\sin m\pi} \prod_{k=1}^{r-2} \frac{1 - b^{k-m}}{1 - b^k} \geq I(ab^{r-1}, b) \geq \frac{\pi}{\sin m\pi} \prod_{k=1}^{r+s-1} \frac{1 - b^{k-m}}{1 - b^k}.$$

Letting r tend to ∞ , we readily deduce (12), and so the proof of (1) is complete.

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