THE FIXED POINT PROPERTY FOR CONTINUA APPROXIMATED FROM WITHIN BY PEANO CONTINUA WITH THIS PROPERTY

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ABSTRACT. Let X be a continuum that is approximated from within by Peano subcontinua with the fixed point property (FPP). Then we show a sufficient condition that X has FPP. As a consequence we have that the Cartesian product of n Warsaw circles is a T^n -like continuum with FPP and the n -fold suspension of Warsaw circle is an S^n -like continuum with FPP.

1. Introduction. Let X be a space that is approximated from within by subsets with the fixed point property (abbreviated FPP), i.e., containing a monotone increasing sequence $C_1 \subset C_2 \subset \cdots$ of subsets with FPP such that $X = \bigcup_i C_i$. Then it is natural to ask when X has FPP. G. S. Young [7] proved that every arcwise connected Hausdorff space in which every monotone collection of arcs is contained in an arc has FPP. (Subsequently he mentioned that in this result compactness is not required [8].) L. E. Ward, Jr. [5] generalized both Young's theorem above and Borsuk's theorem [2] that an arcwise connected hereditarily unicoherent metric curve has FPP.

In this paper we show that if X is approximated from within by Peano subcontinua with FPP and satisfies certain conditions, then it has FPP. As a consequence we have that for every positive integer n the Cartesian product of n Warsaw circles is a T^n -like continuum with FPP and the n -fold suspension of Warsaw circle is an S^n -like continuum with FPP, where T^n and S^n mean an n -dimensional torus and an n -sphere, respectively. Here we refer to the fact that E. Dyer [4] proved that the Cartesian product of n chainable continua has FPP.

2. Main theorem.

DEFINITION. A *continuum* is a compact connected metric space and a *Peano continuum* is a locally connected continuum. A *map* is a continuous function. Let (M, d) be a metric space and ε a positive number. A map $f: M \rightarrow M$ is said to be ε -near to the identity map or simply to be ε -near if $d(x, f(x)) < \varepsilon$ for every $x \in M$.

THEOREM 1. *Let X be a continuum for which there exists a sequence $C_1 \subset C_2 \subset \cdots$ of Peano continua such that $X = \bigcup_i C_i$ and every C_i has FPP. If the following two statements hold, then X has FPP.*

(1) *For every $\varepsilon > 0$, there exists a C_i and a function $f: X \rightarrow X$ such that for each $s > i$ the restriction $f|_{C_s}$ is an ε -near map of C_s to C_i .*

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(2) *There exists a closed subset A , which may be empty, of $\bigcap_j \overline{X - C_j}$, such that every f in (1) is continuous on A , $f(A) \subset C_1$ and every point $x \in \bigcap_j \overline{X - C_j} - A$ has a neighborhood U whose component containing x lies in a C_l .*

PROOF. Let $g: X \rightarrow X$ be any map.

(i) For every C_i and every $\delta > 0$, there exists a C_s ($s > i$) with $g(C_i) - N_\delta(A) \subset C_s$, where $N_\delta(A)$ is a δ -neighborhood of A in X . For otherwise, there are a C_i and a positive number δ such that for every $s > i$, $\{g(C_i) - N_\delta(A)\} \cap (X - C_s)$ contains a point x_s . By compactness of $g(C_i) - N_\delta(A)$, and by taking a convergent subsequence if necessary, we may assume that $\lim x_s = x$ for a point x belonging to both $g(C_i) - N_\delta(A)$ and $\bigcap_s \overline{X - C_s}$. Then we can find U and C_l satisfying condition (2). Since C_i is a Peano continuum, so is $g(C_i)$ by continuity of g . Hence there exists a neighborhood V of x in X such that $g(C_i) \cap V$ is contained in the component K of U with $x \in K$. Since $\{x_s\}$ converges to x , a point x_s ($s > l$) lies in both $\{g(C_i) - N_\delta(A)\} \cap V$ and $X - C_s$. Since $\{g(C_i) - N_\delta(A)\} \cap V \subset K$ and $X - C_s \subset X - C_l$, it follows that $K \cap (X - C_l) \neq \emptyset$, which contradicts (2). Hence the assertion (i) holds.

(ii) For every C_i , $fg(C_i) \subset C_i$ holds, where f is the function in (1). For, using (i) and noting that f carries C_s into C_i , we have $f(g(C_i) - N_\delta(A)) \subset C_i$ for every $\delta > 0$. Since A is closed, we have $\bigcap_{\delta > 0} N_\delta(A) = A$. Hence

$$C_i \supset \bigcup_{\delta > 0} f(g(C_i) - N_\delta(A)) = f\left(g(C_i) - \bigcap_{\delta > 0} N_\delta(A)\right) = f(g(C_i) - A).$$

Since $f(A) \subset C_1$ by our assumption and $C_1 \subset C_i$, we get $fg(C_i) \subset C_i$.

(iii) The map g leaves a point of X fixed. To show this, we must show that the function fg is continuous on C_i . By (i) $g(C_i) - N_\delta(A) \subset C_s$ holds for every $\delta > 0$ and every $s > i$. Since f is continuous on C_s by (1), so is f on $g(C_i) - N_\delta(A)$, and so is f on $g(C_i) - A$. By (2) f is continuous on A . Thus f is continuous on $g(C_i)$.

Now let ε be any positive number, and C_i a set as in (1). Since C_i has FPP, there exists a point $y \in C_i$ with $fg(y) = y$. Thus we have $d(y, g(y)) = d(fg(y), g(y)) < \varepsilon$, because f is ε -near. Since ε is arbitrary and X is compact, g leaves a point of X fixed. $\square$

REMARK 1. An n -sphere S^n ($n \geq 1$) satisfies condition (1) in Theorem 1 but has no FPP.

REMARK 2. In the above theorem, if a point $x \in X$ is not in $\bigcap_j \overline{X - C_j}$, then X is locally connected at x . However, if x is in $\bigcap_j \overline{X - C_j} - A$, then X is not locally connected at x .

3. Cartesian products. It is well known that even if both X_1 and X_2 have FPP, the Cartesian product $X_1 \times X_2$ does not necessarily have FPP (cf. [3]). For continua such as in Theorem 1, we have

THEOREM 2. *Suppose that for each k ($1 \leq k \leq n$), X_k and $C_{k1} \subset C_{k2} \subset \cdots$ satisfy the conditions in Theorem 1. If every $C_{1i} \times C_{2i} \times \cdots \times C_{ni}$ ($i = 1, 2, \dots$) has FPP, then so does $X_1 \times X_2 \times \cdots \times X_n$.*

PROOF. It is sufficient to show that $X = X_1 \times \cdots \times X_n$ and $C_i = C_{1i} \times \cdots \times C_{ni}$ ($i = 1, 2, \dots$) satisfy the conditions in Theorem 1. Obviously C_i are Peano continua in the continuum X such that $X = \bigcup_i C_i$. Now let k be any integer in $\{1, 2, \dots, n\}$.

Condition (1). For every $\varepsilon > 0$, there exist a C_{ki} and a function $f_k: X_k \rightarrow X_k$ such that for each $s > i$ the restriction $f_k|_{C_{ks}}$ is an $\varepsilon/\sqrt{n}$ -near map of C_{ks} to C_{ki} . Let f be the function of X to itself defined by $f(x) = (f_1(x_1), f_2(x_2), \dots, f_n(x_n))$ for $x = (x_1, x_2, \dots, x_n)$. Then the restriction $f|_{C_s}$ is an ε -near map of C_s to C_i under the metric $(d_1^2 + \dots + d_n^2)^{1/2}$ on X , where d_k is a metric on X_k .

Condition (2). By a simple argument we have the equality $\bigcap_j \overline{X - C_j} = \bigcup_k X_1 \times \dots \times X_{k-1} \times L_k \times X_{k+1} \times \dots \times X_n$, where $L_k = \bigcap_j \overline{X_k - C_{kj}}$. Now by (2) in Theorem 1 there exists a closed subset A_k of L_k such that $f_k(A_k) \subset C_{k1}$ and every $y_k \in L_k - A_k$ has a neighborhood U_k whose component containing y_k lies in a C_{kl} . Denote $A_1 \times \dots \times A_n$ by A . Since A_k ($1 \leq k \leq n$) is compact, A is closed in $\bigcap_j \overline{X - C_j}$. Moreover, f is continuous on A and $f(A) = f_1(A_1) \times \dots \times f_n(A_n) \subset C_{11} \times \dots \times C_{n1} = C_1$.

Let $x = (x_1, \dots, x_n)$ be a point in $\bigcap_j \overline{X - C_j} - A$. Then an x_k is not in A_k . If $x_k \in X_k - L_k$, then x_k has a connected neighborhood U_k in X_k contained in a C_{kl_k} . (See Remark 2 in the previous section.) If $x_k \in L_k - A_k$, then there exists a neighborhood U_k of x_k in X_k whose component containing x_k lies in a C_{kl_k} for some l_k . Then $U = U_1 \times \dots \times U_n$ is a neighborhood of x in X whose component containing x lies in $C_l = C_{1l} \times \dots \times C_{nl}$, where $l = \max_k l_k$.

Thus X and C_i satisfy the conditions in Theorem 1, and our proof is complete. $\square$

DEFINITION. Let X and Y be compact metric spaces. Then X is said to be Y -like if for every $\varepsilon > 0$ there is a map f of X onto Y such that for every $y \in Y$ the diameter of $f^{-1}(y)$ is less than ε .

COROLLARY 1. For every positive integer n , the Cartesian product of n Warsaw circles is a T^n -like continuum with FPP.

PROOF. Let X be the Warsaw circle (the $\sin(1/x)$ circle). There is a monotone increasing sequence of arcs C_i such that $X = \bigcup_i C_i$ and $\bigcap_j \overline{X - C_j}$ is the limiting arc. Then X, C_i satisfy the conditions of Theorem 1, in which $A = \emptyset$. Note that if P_i is Q_i -like ($i = 1, 2$), then $P_1 \times P_2$ is $(Q_1 \times Q_2)$ -like. Hence by Theorem 2, the Cartesian product of n Warsaw circles is a T^n -like continuum with FPP. $\square$

REMARK 3. The Cartesian product $X = X_1 \times X_2 \times \dots$ of a countable number of Warsaw circles X_i is a T^ω -like continuum with FPP. For let $(x_1, x_2, \dots)$ be a constant point of X , and define a map $f_i: X \rightarrow X_1 \times \dots \times X_i \times (x_{i+1}) \times \dots$ by $f_i(y_1, \dots, y_i, y_{i+1}, \dots) = (y_1, \dots, y_i, x_{i+1}, \dots)$. Then for every $\varepsilon > 0$ there exists an f_i which is an ε -near map of X to $X_1 \times \dots \times X_i \times (x_{i+1}) \times \dots$ with FPP. Hence X has FPP.

REMARK 4. The Cartesian product of the above T^n -like continuum and an m -cell ($1 \leq m \leq \omega$) has FPP, because every m -cell satisfies the conditions of Theorem 1.

4. Cones and suspensions.

DEFINITION. The cone $Z^\#$ over a topological space Z is the quotient space of $Z \times [0, 1]$ in which $Z \times 0$ is identified to one point a , the vertex of $Z^\#$. The suspension Z^* of Z is the quotient space of $Z \times [-1, 1]$ in which $Z \times (-1)$ is identified to one point a and $Z \times 1$ is identified to another point b . The points a, b are called the suspension points of Z^* .

There is a continuum with FPP, the cone over which has no FPP (cf. [1, p. 129]). Recently T. Watanabe treated FPP for cones over some general spaces (§32 of [6]). For continua such as in Theorem 1, we have

THEOREM 3. *Assume that X and $C_1 \subset C_2 \subset \cdots$ satisfy the conditions in Theorem 1. If every C_i^* ($C_i^\#$) ($i = 1, 2, \dots$) has FPP, then so does the suspension of X (the cone over X).*

We shall prove only the suspension case, as the cone case is similar. The following properties of suspension are readily seen.

LEMMA. *Let P , Q and P_λ be subsets of a topological space X , and S the set consisting of two suspension points of X^* . Then the following hold.*

- (1) $\bigcap_\lambda P_\lambda^* - S = (\bigcap_\lambda P_\lambda)^* - S$.
- (2) $(P - Q)^* - S = P^* - Q^*$.
- (3) $\overline{P^*} = (\overline{P})^*$.
- (4) $\overline{P^* - Q^*} = \overline{P - Q}^*$.

PROOF OF THEOREM 3. Let a, b be the suspension points. A point of X^* is expressible as $(x, t) \in X \times [-1, 1]$, where $(x, -1) = a$, $(x, 1) = b$ for any $x \in X$. Note that for a metric d on X there is a metric d^* on X^* so that $d^*((x, t), (y, t)) \leq (1 - |t|)d(x, y)$ for every x, y in X and $t \in [-1, 1]$.

The sets C_i^* are Peano continua in the continuum X^* such that $X^* = \bigcup_i C_i^*$. Therefore, to prove the theorem it is sufficient to show that conditions (1) and (2) in Theorem 1 hold.

Condition (1). Let ε be any positive number. Then there exist a C_i and a function $f: X \rightarrow X$ as in (1) of Theorem 1. If we define $f^*: X^* \rightarrow X^*$ by $f^*(x, t) = (f(x), t)$, then every restriction $f^*|_{C_s^*}: C_s^* \rightarrow C_i^*$ is ε -near, since $d^*((x, t), (y, t)) \leq d(x, y)$.

Condition (2). Let A be as in (2) of Theorem 1. Then $A^* \subset \{\bigcap_j \overline{X - C_j}\}^* = \bigcap_j \overline{X^* - C_j^*}$ and A^* is closed, by (1) and (2) of the lemma. From $f(A) \subset C_1$, it follows that $f^*(A^*) \subset C_1^*$. Also it is easily seen that f^* is continuous on A^* . By the lemma the set $\bigcap_j \overline{X^* - C_j^*} - A^*$ is equal to $(\bigcap_j \overline{X - C_j} - A)^* - \{a, b\}$. Let (x, t) be any point of this set. Then $x \in \bigcap_j \overline{X - C_j} - A$ and $-1 < t < 1$. Therefore there exists a neighborhood U of x whose component K containing x lies in a C_l . Then $U^* - \{a, b\}$ is a neighborhood of (x, t) , and its component containing (x, t) is $K^* - \{a, b\}$, which lies in C_l^* . Thus we complete our proof. $\square$

COROLLARY 2. *For every positive integer n , the n -fold suspension of a Warsaw circle is an S^n -like continuum with FPP.*

PROOF (INDUCTION ON n). We first note that if P is Q -like, then P^* is Q^* -like. Since the Warsaw circle is an S^1 -like continuum with FPP, its suspension X is a S^2 -like continuum with FPP by Theorem 3. Again applying the theorem to X , in which A is the set of two suspension points, we have an S^3 -like continuum with FPP. Inductively, the $(n - 1)$ -fold suspension of the Warsaw circle is a desired continuum. $\square$

REMARK 5. The Cartesian product of an m -cell ($1 \leq m \leq \omega$) and the above S^n -like continuum has FPP.

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