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REALIZING DIAGRAMS IN THE HOMOTOPY CATEGORY BY MEANS OF DIAGRAMS OF SIMPLICIAL SETS[1]

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ABSTRACT. Given a small category $\mathbf{D}$, we show that a $\mathbf{D}$-diagram $\overline{X}$ in the homotopy category can be realized by a $\mathbf{D}$-diagram of simplicial sets iff a certain simplicial set $r\overline{X}$ is nonempty. Moreover, this simplicial set $r\overline{X}$ can be expressed as the homotopy inverse limit of simplicial sets whose homtopy types are quite well understood. There is also an associated obstruction theory. In the special case that $\mathbf{D}$ is a group (i.e. $\mathbf{D}$ has only one object and all its maps are invertible) these results reduce to the ones of G. Cooke.

1. Introduction.

1.1. *Summary.* Let $\mathbf{D}$ be a small category. The aim of this note then is to obtain *necessary and sufficient conditions in order that a* $\mathbf{D}$*-diagram* $\overline{X}$ *in the homotopy category can be realized by means of a* $\mathbf{D}$*-diagram of simplicial sets.*

This is done by constructing a simplicial set $r\overline{X}$, which is nonempty iff $\overline{X}$ can be so realized. To get a hold on the homotopy type of $r\overline{X}$, one notes that $r\overline{X}$ is the homotopy fibre of a map between a simplicial set $\pi^{-1}c\overline{X}$ which classifies certain $\mathbf{D}$-diagrams of simplicial sets associated with $\overline{X}$ and which was studied in [**5**], and a simplicial set $c\overline{X}$ which classifies similar $\mathbf{D}$-diagrams in the homotopy category. As the homotopy types of $\pi^{-1}c\overline{X}$ and $c\overline{X}$ can be expressed as homotopy inverse limits of simplicial sets whose homotopy types are quite well understood, the same can therefore be done for $r\overline{X}$.

The existence of realizations is also equivalent to the existence of *liftings* of certain maps between diagrams of simplicial sets and there is an associated *obstruction* theory. In the special case that $\mathbf{D}$ is a group (i.e. $\mathbf{D}$ has only one object and all maps of $\mathbf{D}$ are invertible) these results reduce to the ones of G. Cooke [**2**].

1.2. *Notation, terminology, etc.* We will rely heavily on the notation, terminology and results in [**5**]. In particular:

(i) *The division of a category.* Let $\mathbf{D}$ be a category and, for every integer $n \geq 0$, let $\mathbf{n}$ denote the category with the integers $0, \ldots, n$ as objects and with exactly one map $i \to j$ whenever $i \leq j$. The *division* of $\mathbf{D}$ then is the category $d\mathbf{D}$ which has as objects the functors $\mathbf{n} \to \mathbf{D}$ $(n \geq 0)$ and in which the maps $(J_1\colon \mathbf{n}_1 \to \mathbf{D}) \to (J_2\colon \mathbf{n}_2 \to \mathbf{D})$ are the commutative diagrams:

$$\begin{tikzcd} \mathbf{n}_2 \arrow[rr] \arrow[dr, "J_2"'] & & \mathbf{n}_1 \arrow[dl, "J_1"] \\ & \mathbf{D} & \end{tikzcd}$$

(ii) *Simplicial sets.* The category of simplicial sets will be denoted by $\mathbf{S}$. Many of the simplicial sets used in this note are nerves of categories which are not necessarily

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small, but are readily verified to be *homotopically small* in the sense of [**3**, §2]. As explained there, this does not really matter, i.e. one can "do homotopy theory" with them as usual.

(iii) *Categories of diagrams.* If $\mathbf{C}$ is a category and $\mathbf{D}$ is a small category, then we denote by $\mathbf{C}^{\mathbf{D}}$ the category of $\mathbf{D}$-diagrams in $\mathbf{C}$ (which has as objects the functors $\mathbf{D} \to \mathbf{C}$ and as maps the natural transformations between them). The diagram category $\mathbf{S}^{\mathbf{D}}$ admits a *closed simplicial model category* structure in which the simplicial structure is the obvious one and in which a map $X \to Y \in \mathbf{S}^{\mathbf{D}}$ is a weak equivalence or a fibration iff, for every object $D \in \mathbf{D}$, the induced map $XD \to YD \in \mathbf{S}$ is so.

(iv) *Homotopy inverse limits.* As [**1**, Chapter XI] homotopy inverse limits only have homotopy meaning when applied to fibrant diagrams, we often have to replace a given diagram $Y \in \mathbf{S}^{\mathbf{D}}$ by a weakly equivalent fibrant one such as, for instance, $\mathrm{Ex}^\infty Y$, where Ex^∞ denotes the functor of [8]. To simplify the notation *we will write Y^f instead of $\mathrm{Ex}^\infty Y$.*

(v) *The homotopy category.* This is the category $\mathrm{ho}\,\mathbf{S}$ obtained from $\mathbf{S}$ by localizing with respect to (i.e. formally inverting) the weak equivalences. The resulting projection functor will be denoted by $\pi\colon \mathbf{S} \to \mathrm{ho}\,\mathbf{S}$ and we will use the same symbol for the induced functors $\pi\colon \mathbf{S}^{\mathbf{D}} \to (\mathrm{ho}\,\mathbf{S})^{\mathbf{D}}$.

(vi) *We do not distinguish in notation between a small category and its nerve.*

2. A classification result for diagrams. In preparation for our realization results (§3) we investigate here the following.

2.1. *Classification problem for diagrams.* Given a category $\mathbf{C}$ and a small category $\mathbf{D}$, call (in the notation of 1.2) two objects $\overline{X}, \overline{Y} \in \mathbf{C}^{\mathbf{D}}$ *conjugate* if, for every integer $n \geq 0$ and every functor $J\colon\ \mathbf{n} \to \mathbf{D}$, the induced $\mathbf{n}$-diagrams $J^*\overline{X}$ and $J^*\overline{Y}$ are isomorphic. The problem then is, given a diagram $\overline{X} \in \mathbf{C}^{\mathbf{D}}$ to *classify the isomorphism classes of the conjugates of $\overline{X}$*. Of course one can do this, in a rather trivial manner, by means of

2.2. *The classification complex $c\overline{X}$ of a diagram $\overline{X} \in \mathbf{C}^{\mathbf{D}}$*. This is the nerve (1.2(ii)) of the subcategory of $\mathbf{C}^{\mathbf{D}}$ which consists of the conjugates of X and all isomorphisms between them.

Clearly this definition implies

2.3. PROPOSITION. *Let $\mathbf{C}$ be a category, $\mathbf{D}$ a small category and $\overline{X} \in \mathbf{C}^{\mathbf{D}}$. Then*

(i) *there is an obvious* 1-1 *correspondence between the isomorphism classes of the conjugates of $\overline{X}$ and the components of $c\overline{X}$, and*

(ii) *for every conjugate $\overline{Y}$ of $\overline{X}$, the corresponding* (*see* (i)) *component of $c\overline{X}$ has the homotopy type of $K(\mathrm{aut}\,\overline{Y}, 1)$, where $\mathrm{aut}\,\overline{Y}$ denotes the group of automorphisms of $\overline{Y}$.*

To get a better hold on the homotopy type of the whole classification complex $c\overline{X}$ one needs

2.4. *The classification diagram $c_{d\mathbf{D}}\overline{X}$ of a diagram $\overline{X} \in \mathbf{C}^{\mathbf{D}}$*. This is the $d\mathbf{D}$-diagram (1.2((i)) which consists of the classification complexes $cJ^*\overline{X}$, where J runs through all functors $\mathbf{n} \to \mathbf{D}$ ($n \geq 0$).

The main result of this section is given in Theorem 2.5.

2.5. THEOREM. *Let* $\overline{X} \in \mathbf{C}^{\mathbf{D}}$. *Then the natural map* [**1**, *Chapter* XI, §4]

$$c\overline{X} = \varprojlim^{d\mathbf{D}} c_{d\mathbf{D}}\overline{X} \to \operatorname{ho}\varprojlim^{d\mathbf{D}} c_{d\mathbf{D}}\overline{X}$$

is a weak equivalence.

2.6. COROLLARY [**1**, CHAPTER XI, 7.2]. *The components of* $c\overline{X}$ *are in a natural* 1-1 *correspondence with the elements of the set* $\varprojlim^1(\pi_1 c_{d\mathbf{D}}\overline{X})$.

2.7. REMARK. Theorem 2.5 has the same form as Theorem 3.4(iii) of [**5**] and hence admits the same *variations* [**5**, §§3-5].

The proof of Theorem 2.5 uses the following lemma, the proof of which is lengthy but straightforward and will be left to the reader.

2.8. LEMMA. *Let* $\mathbf{E}$ *be a small category which is inverse* (*i.e.* $\mathbf{E}^{\mathrm{op}}$ *is direct* [**4**, §4]). *Then the category* $\mathbf{S}^{\mathbf{E}}$ *admits a closed model category structure in which a map* $X \to Y \in \mathbf{S}^{\mathbf{E}}$ *is a weak equivalence or a cofibration iff, for every object* $E \in \mathbf{E}$, *the induced map* $XE \to YE \in \mathbf{S}$ *is so. Moreover, if* $Y \in \mathbf{S}^{\mathbf{E}}$ *is fibrant with respect to this model category structure, then* Y *is fibrant in the usual sense* (1.2(iii)) *and the natural map* $\varprojlim^{\mathbf{E}} Y \to \operatorname{ho}\varprojlim^{\mathbf{E}} Y$ *is a weak equivalence.*

PROOF OF THEOREM 2.5. We first deal with the case that $\mathbf{D}$ is a *direct* [**4**, §4] category. Let $sd\mathbf{D}$ be the subdivision of $\mathbf{D}$, i.e. [**4**, §5] the category obtained from $d\mathbf{D}$ by turning all "degeneracy maps" (i.e. triangles as in 1.2(i) in which the top map is onto) into identity maps. As $\mathbf{D}$ was assumed to be direct, the projection $s\colon d\mathbf{D} \to sd\mathbf{D}$ admits an obvious cross section $t\colon sd\mathbf{D} \to d\mathbf{D}$. Moreover, $sd\mathbf{D}$ is an inverse category and hence (2.8) the natural map

$$c\overline{X} = \varprojlim^{sd\mathbf{D}} t^* c_{d\mathbf{D}}\overline{X} \to \operatorname{ho}\varprojlim^{sd\mathbf{D}} t^* c_{d\mathbf{D}}\overline{X}$$

is a weak equivalence. The desired result (for $\mathbf{D}$ direct) now follows readily from [**5**, 6.6, 6.11 and 9.3].

To complete the proof one uses essentially the arguments of [**5**, 8.1].

3. The realization results. We start with formulating

3.1. *The realization problem.* Given a small category D and a diagram $\overline{X} \in (\operatorname{ho}\mathbf{S})^{\mathbf{D}}$ (1.2(v)), define a *realization* of $\overline{X}$ as a pair (Y, f) such that Y is an object of $\mathbf{S}^{\mathbf{D}}$ and f is an isomorphism $f\colon \pi Y \xrightarrow{\approx} \overline{X} \in (\operatorname{ho}\mathbf{S})^{\mathbf{D}}$ (1.2(v)) and define a *weak equivalence* between two such realizations (Y, f_Y) and (Z, f_Z) as a weak equivalence $g\colon Y \to Z \in \mathbf{S}^{\mathbf{D}}$ (1.2(iii)) such that $(\pi g)f_Z = f_Y$. Our realization problem then is to find *necessary and sufficient conditions in order that* X *have a realization* and *to classify the weak equivalence classes of such realizations.*

An obvious solution to this problem is provided by

3.2. *The realization complex* $r\overline{X}$ *of a diagram* $\overline{X} \in (\text{ho. }\mathbf{S})^{\mathbf{D}}$. This is defined as the nerve (see 1.2(ii)) of the full subcategory of the over category $\pi \downarrow \overline{X}$ (1.2(v)) generated by the realizations of $\overline{X}$, i.e. the pairs (Y, f) for which the map $f\colon \pi Y \to \overline{X} \in (\operatorname{ho}\mathbf{S})^{\mathbf{D}}$ is an isomorphism.

An immediate consequence of this definition is

3.3. THEOREM. *Let* $\overline{X} \in (\operatorname{ho}\mathbf{S})^{\mathbf{D}}$. *Then*

(i) $\overline{X}$ *can be realized iff the realization complex* $r\overline{X}$ *is nonempty, and*

(ii) *there is an obvious* 1-1 *correspondence between the weak equivalence classes of the realizations of* $\overline{X}$ *and the components of* $r\overline{X}$ *containing them.*

To get some hold on the homotopy type of $r\overline{X}$, let $c\overline{X}$ be as in 2.2 and let, as in [**5**, 1.4], $\pi^{-1}c\overline{X}$ denote the nerve of the subcategory of $\mathbf{S}^{\mathbf{D}}$ which consists of the diagrams $Y \in \mathbf{S}^{\mathbf{D}}$ such that $\pi Y \in (\mathrm{ho}\,\mathbf{S})^{\mathbf{D}}$ is conjugate (2.1) to $\overline{X}$, and all weak equivalences between them. Theorem B of Quillen [**10**] then readily implies

3.4. PROPOSITION. *Let* $\overline{X} \in (\mathrm{ho}\,\mathbf{S})^{\mathbf{D}}$. *Then* $r\overline{X}$ *is the homotopy fibre of the projection map* $\pi^{-1}c\overline{X} \to c\overline{X}$, *over the component of* $c\overline{X}$ *containing* $\overline{X}$.

To describe the homotopy types of the components of $r\overline{X}$ let, for a diagram $Z \in \mathbf{S}^{\mathbf{D}}$ which is both fibrant and cofibrant (1.2(iii)), haut Z be its simplicial monoid of self-weak equivalences [**5**, §2] and let $\mathrm{haut}_0\, Z \subset \mathrm{haut}\, Z$ be its simplicial monoid of *restricted self-weak equivalences,* i.e. the maximal simplicial submonoid which has as its vertices the self-weak equivalences $Z \to Z$ which, for every object $D \in \mathbf{D}$, induce a map $ZD \to ZD \in \mathbf{S}$ that is homotopic to the identity. An easy consequence of 3.4 and [**5**, 2.3] then is

3.5. THEOREM. *Let* $\overline{X} \in (\mathrm{ho}\,\mathbf{S})^{\mathbf{D}}$ *and let* (Y, f) *be a realization of* $\overline{X}$. *Then the component of* $r\overline{X}$ *containing* (Y, f) *has the homotopy type of "a classifying complex for the restricted self weak-equivalences of* Y*," i.e. the homotopy type of the classifying complex* [**9**, *p.* 87]. $\overline{W}\,\mathrm{haut}_0\, Z$, *where* $Z \in \mathbf{S}^{\mathbf{D}}$, *is any diagram which is fibrant and cofibrant and weakly equivalent to* Y.

To express the homotopy type of the whole realization complex $r\overline{X}$ in terms of more easily accessible simplicial sets we need

3.6. *The realization diagram* $r_{d\mathbf{D}}\overline{X}$ *of a diagram* $\overline{X} \in (\mathrm{ho}\,\mathbf{S})^{\mathbf{D}}$. This is the $d\mathbf{D}$-diagram (1.2(i)) which consists of the realization complexes $rJ^*\overline{X}$, where J runs through all functors $\mathbf{n} \to \mathbf{D}$ $(n \geq 0)$.

Using 2.5, 3.4, [**5**, 3.4(iii)] and [**1**, Chapter XI, 5.5] one then readily proves

3.7. THEOREM. *Let* $\overline{X} \in (\mathrm{ho}\,\mathbf{S})^{\mathbf{D}}$. *Then the natural map* (1.2(iv) *and* (vi))

$$r\overline{X} = \varprojlim{}^{d\mathbf{D}}\, r_{d\mathbf{D}}\overline{X} \to \mathrm{ho}\,\varprojlim{}^{d\mathbf{D}}(r_{d\mathbf{D}}\overline{X})^f = \hom^{d\mathbf{D}}((dD \downarrow -), (r_{d\mathbf{D}}\overline{X})^f)$$

is a weak equivalence.

Moreover, one has as in [**5**, 5.1]

3.8. VARIATION. *Let* $\overline{X} \in (\mathrm{ho}\,\mathbf{S})^{\mathbf{D}}$, *let* $g\colon d\mathbf{D} \to \mathbf{E}$ *be a functor between small categories and let* $u \in \mathbf{S}^{\mathbf{E}}$ *be a fibrant diagram such that* $r_{d\mathbf{D}}\overline{X}$ *is weakly equivalent to the pull back diagram* g^*u. *Then the realization complex* $r\overline{X}$ *is weakly equivalent to the function complex* (1.2(vi)) $\hom^{\mathbf{E}}((g \downarrow -), u)$.

To show that *the realization problem is equivalent to a lifting problem* denote by $v\colon (d\mathbf{D} \downarrow -) \to c_{d\mathbf{D}}\overline{X}$ the vertex of $\hom^{d\mathbf{D}}((dD \downarrow -), c_{d\mathbf{D}}\overline{X}) = \mathrm{ho}\,\varprojlim{}^{d\mathbf{D}}\, c_{d\mathbf{D}}\overline{X}$ for $\overline{X} \in (\mathrm{ho}\,\mathbf{S})^{\mathbf{D}}$, which corresponds to the vertex of $c\overline{X} = \varprojlim{}^{d\mathbf{D}}\, c_{d\mathbf{D}}\overline{X}$ given by $\overline{X}$ itself, let $\pi^{-1}c_{d\mathbf{D}}\overline{X}$ denote the $d\mathbf{D}$-diagram which consists of the simplicial sets $\pi^{-1}cJ^*\overline{X}$, where J runs through all functors $\mathbf{n} \to \mathbf{D}$ $(n \geq 0)$ and let

$$\pi^{-1}c_{d\mathbf{D}}\overline{X} \xrightarrow{\sim} (\pi^{-1}c_{d\mathbf{D}}\overline{X})' \xrightarrow{\pi'} c_{d\mathbf{D}}\overline{X}$$

be a factorization of the projection into a trivial cofibration and a fibration π'.

Then 2.5 and 4.2 imply

3.9. THEOREM. *Let $\overline{X} \in (\mathrm{ho}\,\mathbf{S})^{\mathbf{D}}$. Then $\overline{X}$ has a realization iff there exists a dotted arrow which makes the following diagram commutative*:

$$\begin{tikzcd} & (\pi^{-1} c_{d\mathbf{D}} \overline{X}) \arrow[d, "\pi'"] \\ (dD \downarrow -) \arrow[r, "v"] \arrow[ur, dashed] & c_{d\mathbf{D}} \overline{X} \end{tikzcd}$$

There are, of course, associated

3.10. OBSTRUCTIONS. The lifting problem of 3.9 is of the form considered in [**7**, 3.7]. The obstruction theory outlined there thus applies, with obstruction cocycles lying in the groups $Z^{n+1}((d\mathbf{D} \downarrow -); \Pi_n(\pi^{-1} c_{d\mathbf{D}} \overline{X}))$, $n \geq 2$.

3.11. REMARK. One can generalize 3.9 and 3.10 in the same manner as 3.7 was generalized to 3.8. For example, if $\overline{X} \in (\mathrm{ho}\,\mathbf{S})^{\mathbf{D}}$ and $\mathbf{D}$ is a *group* (i.e. $\mathbf{D}$ has only one object D and all maps in $\mathbf{D}$ are invertible), then, using the argument of [**5**, 5.3], one readily recovers the result of G. Cooke [**2**] that *the realization problem for $\overline{X}$ is equivalent to the lifting problem*

$$\begin{tikzcd} & (\overline{W}\,\mathrm{haut}\, Y)^f \arrow[d] \\ \mathbf{D} \arrow[r] \arrow[ur, dashed] & K(\pi_0\,\mathrm{haut}\, Y, 1) \end{tikzcd}$$

where Y is any fibrant simplicial set such that $\pi Y = \overline{X} D \in \mathrm{ho}\,\mathbf{S}$.

We end with observing

3.12. REMARK. Using the results of [**6**] one can obtain similar realization results for diagrams indexed by small *simiplicial* categoriės.

BIBLIOGRAPHY

1. A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math., vol. 304, Springer-Verlag, Berlin and New York, 1972.
2. G. Cooke, *Replacing homotopy actions by topological actions*, Trans. Amer. Math. Soc. **237** (1978), 391–406.
3. W. G. Dwyer and D. M. Kan, *Function complexes in homotopical algebra*, Topology **19** (1980), 427–440.
4. ——, *Function complexes for diagrams of simplicial sets*, Nederl. Akad. Wetensch. Proc. Ser. A **86** = Indag. Math. **45** (1983).
5. ——, *A classification theorem for diagrams of simplicial sets*, Topology (to appear).
6. ——, *Equivariant homotopy classification*, J. Pure Appl. Algebra (to appear).
7. ——, *An obstruction theory for diagrams of simplicial sets*, Nederl. Akad. Wetenesch. Proc. Ser. A **87** = Indag. Math. **46** (1984).
8. D. M. Kan, *On c.s.s. complexes*, Amer. J. Math. **79** (1957), 449–476.
9. J. P. May, *Simplicial objects in algebraic topology*, Van Nostrand, Princeton, N.J., 1967.
10. D. G. Quillen, *Higher algebraic K-theory*. I, Lecture Notes in Math., vol. 341, Springer-Verlag, Berlin and New York, 1973, pp. 85–147.

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