A CHARACTERIZATION OF M_0 -SPACES

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ABSTRACT. The following result is obtained.

THEOREM. A T_1 -space has a σ -closure-preserving base consisting of clopen sets iff the space can be embedded in the product of countably many σ -discrete stratifiable spaces.

1. Introduction. In [7], R. W. Heath and the author called a topological space an M_0 -space if the space has a σ -closure-preserving base by clopen (= closed and open) sets. Every M_0 -space is a strongly zero-dimensional M_1 -space. In [13], J. Nagata asked whether every strongly zero-dimensional M_1 -space is an M_0 -space.

G. Gruenhage showed in [6] that every σ -discrete stratifiable space is an M_1 -space and his proof actually established that such a space is an M_0 -space. It is easily seen that the property of being an M_0 -space is hereditary and countably productive. Consequently, every subspace of the product of countably many σ -discrete stratifiable spaces is an M_0 -space; in the following we show that, among T_1 -spaces, all M_0 -spaces can be obtained (up to homeomorphisms) in this manner. A particular case of our result was proved in [9], where it was shown that every strongly zero-dimensional F_σ -metrizable stratifiable space is homeomorphic with a subspace of the product of countably many σ -discrete stratifiable spaces (a topological space is F_σ -metrizable if the space can be covered by countably many closed and metrizable subspaces).

Our topological terminology follows that of [5]. For basic facts concerning stratifiable and semistratifiable spaces, see [3, 1, 4 and 10]. For notation and terminology concerning neighborsets, see [8].

2. The characterization. To prove our theorem, we need the following lemma.

LEMMA. Let (X, τ) be a strongly zero-dimensional, paracompact semistratifiable Hausdorff-space, and let V be a transitive neighborset of (X, τ) . Then there exists a strongly zero-dimensional metrizable topology π on X such that $\pi \subset \tau$ and V is a neighborset of the space (X, π) .

PROOF. By Corollary 4.10 of [8], there exists a closed cover $\mathcal{F} = \bigcup_{n < \omega} \mathcal{F}_n$ of (X, τ) such that for each $n < \omega$, the family $\mathcal{F}_n$ is discrete and, for each $F \in \mathcal{F}$, $VF = V\{x\}$ for every $x \in F$. Note that, for each $n < \omega$, the family

$$\mathcal{V}_n = \{(VF) \sim \bigcup(\mathcal{F}_n \sim \{F\}) : F \in \mathcal{F}_n\} \cup \{X \sim \bigcup \mathcal{F}_n\}$$

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is an open cover of (X, τ) . As a semistratifiable Hausdorff-space, (X, τ) has a G_δ -diagonal. Consequently, there exists a sequence $\langle \mathcal{U}_n \rangle_{n < \omega}$ of open covers of (X, τ) such that for every $x \in X$, we have $\bigcap_{n < \omega} \text{St}(x, \mathcal{U}_n) = \{x\}$. Since (X, τ) is a strongly zero-dimensional paracompact space, every open cover of (X, τ) has a disjoint open refinement. It follows that there exists a sequence $\langle \mathcal{W}_n \rangle_{n < \omega}$ of disjoint open covers of (X, τ) such that, for each $n < \omega$, $\mathcal{W}_{n+1}$ refines $\mathcal{W}_n$ and $\mathcal{W}_n$ refines both $\mathcal{V}_n$ and $\mathcal{U}_n$. Let π be the topology on X that has the family $\bigcup_{n < \omega} \mathcal{W}_n$ as a base. It is easily seen that $\pi \subset \tau$ and that the space (X, π) is strongly zero-dimensional and metrizable. To show that V is a neighbornet of (X, π) , let $x \in X$. Then there exists $n < \omega$ and $F \in \mathcal{F}_n$ such that $x \in F$. It is easy to see that the set $(VF) \sim \bigcup(\mathcal{F}_n \sim \{F\})$ is the only member of $\mathcal{V}_n$ that contains x ; it follows, since $\mathcal{W}_n$ refines $\mathcal{V}_n$, that $\text{St}(x, \mathcal{W}_n) \subset VF$. Since $VF = V\{x\}$ and since $\text{St}(x, \mathcal{W}_n)$ is a neighborhood of x in (X, π) , we have shown that $V\{x\}$ is a neighborhood of x in (X, π) . $\square$

COROLLARY. *Let (X, τ) be a strongly zero-dimensional, paracompact semistratifiable Hausdorff-space, and let $\mathcal{F}$ be a closure-preserving family of closed subsets of (X, τ) . Then there exists a strongly zero-dimensional metrizable topology π on X such that $\pi \subset \tau$ and the family $\mathcal{F}$ is closure-preserving and closed in (X, π) .*

PROOF. Define a transitive neighbornet V of (X, τ) by setting, for each $x \in X$, $V\{x\} = X \sim \bigcup(\mathcal{F} \sim \mathcal{F}_x)$, where $\mathcal{F}_x = \{F \in \mathcal{F} : x \in F\}$. Let π be a topology on X that satisfies the conclusion of the lemma. Note that, for all $x \in X$ and $F \in \mathcal{F}$, if $F \cap V\{x\} \neq \emptyset$, then $x \in F$. It follows, since V is a neighbornet of (X, π) , that the family $\mathcal{F}$ is closure-preserving and closed in (X, π) . $\square$

REMARK. The above results remain valid if all occurrences of the phrase “strongly zero-dimensional” are deleted; in this modified form, the result of the corollary is a slight generalization of a result obtained independently by S. Oka and Ju. H. Bregman (Lemma 3.12 of [14] and Proposition 1 of [2]).

We are now ready to prove our main result.

THEOREM. *A T_1 -space is an M_0 -space iff the space is homeomorphic with a subspace of the product of countably many σ -discrete stratifiable spaces.*

PROOF. It was observed in the introduction that the condition in the theorem is sufficient for a space to be an M_0 -space. To prove the necessity of the condition, let the T_1 -space (X, τ) be an M_0 -space. Let $\mathcal{B} = \bigcup_{n < \omega} \mathcal{B}_n$ be a base for τ such that $\mathcal{B}$ consists of clopen sets and, for each $n < \omega$, the family $\mathcal{B}_n$ is closure-preserving in (X, τ) . Without loss of generality, we can assume that each $\mathcal{B}_n$ is closed under finite intersections. We show that for each $n < \omega$, there exists an F_σ -metrizable M_0 -topology η_n on X such that $\mathcal{B}_n \subset \eta_n \subset \tau$. Let $n < \omega$ be given. Denote by $\mathcal{P}_n$ the partition of X determined by the sets $\bigcap \mathcal{C} \sim \bigcup(\mathcal{B}_n \sim \mathcal{C})$, where $\mathcal{C} \subset \mathcal{B}_n$ (we let $\bigcap \emptyset = X$). It follows from Theorems 3.14 and 4.8 (and Remark 4.8.2°) of [8] that the partition $\mathcal{P}_n$ has a refinement $\bigcup_{k < \omega} \mathcal{F}_{n,k}$ such that each $\mathcal{F}_{n,k}$ is closed and discrete in (X, τ) . Since the supremum of countably many strongly zero-dimensional metrizable topologies on X is a strongly zero-dimensional metrizable topology on X , it follows from the above corollary that there exists a strongly zero-dimensional metrizable topology π_n

on X such that $\pi_n \subset \tau$ and each of the families $\mathcal{B}_n$ and $\mathcal{F}_{n,k}$, $k < \omega$, is closure-preserving and closed in (X, π_n) . Note that for every $k < \omega$, since $\mathcal{F}_{n,k}$ is a closure-preserving and disjoint family of closed subsets of (X, π_n) , $\mathcal{F}_{n,k}$ is a discrete family in (X, π_n) . Denote by η_n the topology on X that has the family $\mathcal{B}_n \cup \pi_n$ as a subbase, and note that $\mathcal{B}_n \subset \eta_n \subset \tau$. We show that η_n is an M_0 -topology. Since π_n is a strongly zero-dimensional metrizable topology, π_n has a base $\bigcup_{k < \omega} \mathcal{D}_{n,k}$, where each $\mathcal{D}_{n,k}$ is a discrete family of clopen subsets of (X, π_n) . Since the families $\mathcal{B}_n$ and $\mathcal{D}_{n,k}$, $k < \omega$, are closure-preserving and closed in (X, π_n) , the families $\mathcal{E}_{n,k} = \{B \cap D : B \in \mathcal{B}_n \text{ and } D \in \mathcal{D}_{n,k}\}$, $k < \omega$, are closure-preserving and closed in (X, π_n) and hence also in (X, η_n) . Clearly, the σ -closure-preserving family $\bigcup_{k < \omega} \mathcal{E}_{n,k}$ is a base for the topology η_n . Since $\bigcup_{k < \omega} \mathcal{E}_{n,k}$ consists of clopen subsets of (X, η_n) , we have shown that (X, η_n) is an M_0 -space. Next we show that (X, η_n) is F_σ -metrizable. For every $k < \omega$, denote by $F_{n,k}$ the set $\bigcup \mathcal{F}_{n,k}$; note that this set is closed in (X, π_n) and hence also in (X, η_n) . Since $\bigcup_{k < \omega} F_{n,k} = X$, the F_σ -metrizability of (X, η_n) follows once we show that, for each $k < \omega$, the restriction $\eta_{n,k}$ of the topology η_n to $F_{n,k}$ is the same as the restriction $\pi_{n,k}$ of the metrizable topology π_n to $F_{n,k}$. Since $\pi_n \subset \eta_n$, we have $\pi_{n,k} \subset \eta_{n,k}$. To show that $\eta_{n,k} \subset \pi_{n,k}$, let $x \in F_{n,k}$ and let G be a neighborhood of x in the topology $\eta_{n,k}$. Then there exist $B \in \mathcal{B}_n$ and $D \in \pi_n$ such that $x \in B \cap D \cap F_{n,k} \subset G$. Let F be that member of $\mathcal{F}_{n,k}$ which contains x . From the definition of $\mathcal{B}_n$ and $\mathcal{F}_{n,k}$, it follows that $F \subset \bigcap \{C \in \mathcal{B}_n : x \in C\} \subset B$; consequently, $F \cap D \subset G$. But the set F is open in $(F_{n,k}, \pi_{n,k})$ because $F \in \mathcal{F}_{n,k}$ and the family $\mathcal{F}_{n,k}$ is a discrete closed cover of $(\mathcal{F}_{n,k}, \pi_{n,k})$. It follows from the foregoing that the set G is a neighborhood of x in $(F_{n,k}, \pi_{n,k})$. We have shown that $\eta_{n,k} \subset \pi_{n,k}$; hence $\eta_{n,k} = \pi_{n,k}$ and the topology $\eta_{n,k}$ is metrizable.

Since $\mathcal{B}_n \subset \eta_n \subset \tau$ for each $n < \omega$, and since $\bigcup_{n < \omega} \mathcal{B}_n$ is a base for τ , we have that $\tau = \sup_{n < \omega} \eta_n$. It follows that (X, τ) is homeomorphic with a subspace of the product space $\prod_{n < \omega} (X, \eta_n)$. By Corollary 3.3 of [9], each one of the spaces (X, η_n) is homeomorphic with a subspace of the product of countably many σ -discrete stratifiable spaces. From the foregoing it follows that (X, τ) is homeomorphic with a subspace of the product of countably many σ -discrete stratifiable spaces. $\square$

REMARK. It follows from the above proof that every M_0 -, T_1 -space is homeomorphic with a closed subspace of the product of countably many F_σ -metrizable stratifiable spaces.

In [12], K. Nagami called a topological space a μ -space if the space can be embedded in the product of countably many paracompact F_σ -metrizable spaces. It follows from the above theorem that every M_0 -, T_1 -space is a μ -space. T. Mizokami has shown that every strongly zero-dimensional stratifiable μ -space is an M_0 -space [11, Corollary 4.7]. Consequently, we have the following result.

PROPOSITION. *A T_1 -space is an M_0 -space iff the space is a strongly zero-dimensional stratifiable μ -space.*

ADDED IN PROOF. The result of the Proposition has been obtained independently by M. Itō [M_0 -spaces are μ -spaces, preprint].

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