ON THE RATIONALITY OF THE VARIETY OF SMOOTH RATIONAL SPACE CURVES WITH FIXED DEGREE AND NORMAL BUNDLE

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ABSTRACT. Let $\tilde{S}_{n,a}$ be the variety of smooth, rational curves of degree n in $\mathbf{P}_3$ whose normal bundle has a factor of degree $2n - 1 + a$ and a factor of degree $2n - 1 - a$. In this paper we prove that $\tilde{S}_{n,a}$ is rational if $n - a$ is even and $a > 0$.

We work over $\mathbf{C}$. Let $\tilde{S}_{n,a} \subset \text{Hilb } \mathbf{P}_3$ be the set of smooth, rational curves in $\mathbf{P}_3$ of degree n whose normal bundle splits with a summand of degree $2n - 1 - a$ and another of degree $2n - 1 + a$. Eisenbud and Van de Ven [1, 2] proved that for $0 \leq a \leq n - 4$, $\tilde{S}_{n,a}$ is not empty, irreducible and of dimension $4n - 2a + 1$ (if $a > 0$). Let $S_{n,a}$ be the set of embeddings $f: \mathbf{P}_1 \rightarrow \mathbf{P}_3$ with $f(\mathbf{P}_1) \in \tilde{S}_{n,a}$. They proved in [2] that $S_{n,a}$ is irreducible, rational and, if $a > 0$, of dimension $4n - 2a + 4$. $\text{PGL}(2) = \text{Aut}(\mathbf{P}_1)$ acts naturally on $S_{n,a}$ without fixed points. $\tilde{S}_{n,a}$ is the quotient of $S_{n,a}$ by this action and the natural map $S_{n,a} \rightarrow \tilde{S}_{n,a}$ makes $S_{n,a}$ a principal locally isotrivial bundle over $\tilde{S}_{n,a}$ with structural group $\text{PGL}(2)$ (see Serre [6] for this notion).

In the introduction to [2] Eisenbud and Van de Ven raised the question of the rationality of $\tilde{S}_{n,a}$. Here we prove the following

THEOREM. *If $a > 0$ and $n - a$ is even, then $\tilde{S}_{n,a}$ is rational.*

The proof of this theorem uses only the construction in [2, §5], elementary properties of conic bundles (or $\mathbf{P}_1$ -bundles) with smooth fibers and smooth base, and the definition of stably rational varieties due to Kollar and Schreyer [4]. An irreducible variety V is said to be stably rational of level k if $V \times \mathbf{P}_k$ is rational. For the elementary properties of conic bundles we need to see Serre [6]; we also found useful [3, 5].

We write $\tilde{S}_n$ for the variety of smooth, rational curves of degree n in $\mathbf{P}_3$ and S_n for the set of embeddings of degree n of $\mathbf{P}_1$ into $\mathbf{P}_3$. S_n is rational and $S_n \rightarrow \tilde{S}_n$ is a principal locally isotrivial bundle with structure group $\text{PGL}(2)$. Since S_n (resp. $S_{n,a}$) is rational, if the natural map $p: S_n \rightarrow \tilde{S}_n$ (resp. $S_{n,a} \rightarrow \tilde{S}_{n,a}$) has a rational section,

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then $\tilde{S}_n$ (resp. $\tilde{S}_{n,a}$) is stably rational of level 3. The rationality of $S_{n,a}$ was proved in [2, p. 97].

LEMMA 1. Assume n odd. Then for every $x \in \tilde{S}_n$, there exists a rational section of p defined at x .

PROOF. Since $\tilde{S}_n$ is contained in $\text{Hilb } \mathbf{P}_3$, we have a universal curve $C \rightarrow \tilde{S}_n$ with an inclusion $i: C \rightarrow \tilde{S}_n \times \mathbf{P}_3$ over $\tilde{S}_n$. C is a conic bundle with a smooth base. Since n is odd, this conic bundle is locally trivial in the Zariski topology [2]. Thus there is a neighborhood U of x and an U -isomorphism $h: U \times \mathbf{P}_1 \rightarrow C$. The map $i \circ h$ gives the section of p defined on U . $\square$

We write R_n for the set of maps of degree n of $\mathbf{P}_1$ into $\mathbf{P}_3$. Again $\text{PGL}(2)$ acts on R_n and we write $\tilde{R}_n$ for its quotient. Since we are interested only at birational geometry, there is no problem here; we can substitute R_n with $\tilde{S}_n$ if we want. In [2] a key point was the map $G: S_{n,a} \rightarrow R_{n-a-1}$ ($a > 0$) constructed in the following way. Fix $f \in S_{n,a}$.

$$N_f := f^*(N_{f(\mathbf{P}_1)/\mathbf{P}_3}) \cong \mathcal{O}_{\mathbf{P}_1}(2n-1-a) \oplus \mathcal{O}_{\mathbf{P}_1}(2n-1+a)$$

is a quotient of $f^*(T\mathbf{P}_3)$. Thus the subline bundle $\mathcal{O}_{\mathbf{P}_1}(2n-1+a)$ defines a rank-2 subbundle V_f of $f^*(T\mathbf{P}_3)$. The map $G(f): \mathbf{P}_1 \rightarrow \mathbf{P}_3$ is constructed by taking for $G(f)(t)$ the plane in $\mathbf{P}_3$ which is determined by $V_{f,t} \subset T\mathbf{P}_{3,f(t)}$. Note that the map G descends to a map $\tilde{G}: \tilde{S}_{n,a} \rightarrow \tilde{R}_{n-a-1}$ such that, for $0 < a \leq n-4$ we have the following commutative diagram:

$$\begin{array}{ccc} S_{n,a} & \xrightarrow{G} & R_{n-a-1} \\ \downarrow q & & \downarrow g \\ \tilde{S}_{n,a} & \xrightarrow{\tilde{G}} & \tilde{R}_{n-a-1} \end{array} \quad 0 < a \leq n-4.$$

Eisenbud and Van de Ven [2, p. 97] proved that G is birationally the projection of a product with fiber rational of dimension $2a+5$. If $n-a$ is even, by Lemma 1 g has a rational section. Thus $\tilde{R}_{n-a-1}$ is stably rational of level 3, $\tilde{G}$ is birationally a product with fiber $\mathbf{P}_{2a+5}$ and $\tilde{S}_{n,a}$ is rational. This concludes the proof of Theorem 1.

If $n-a$ is odd, $a > 0$, we do not know very much. A trick easily gives the following

PROPOSITION 1. Assume $a > 0$. Then $\tilde{S}_{n,a}$ is covered by rational subvarieties of codimension 2.

PROOF. Fix a point $O \in \mathbf{P}_1$ and a point P in $\mathbf{P}_3$. Let A_n be the set of embeddings f of $\mathbf{P}_1$ into $\mathbf{P}_3$ with $f(O) = P$ and $\deg(f(\mathbf{P}_1)) = n$. A_n is rational. The affine group of projective transformations of $\mathbf{P}_1$ fixing O acts on A_n and let $\tilde{A}_n \subset \text{Hilb } \mathbf{P}_3$ be the quotient. A_n is the subset of S_n formed by curves through P . The map $A_n \rightarrow \tilde{A}_n$ has always a rational section. This follows from the speciality of the affine group [3, Lemme 2.3]. Alternatively the restriction to $\tilde{A}_n$ of the conic bundle of Lemma 1 comes from a vector bundle since the point P defines a line bundle on $p^{-1}(\tilde{A}_n)$ with degree one on every fiber.

Thus $\tilde{A}_{n-a-1}$ is stably rational of level 2 and $\tilde{G}|_{\tilde{G}^{-1}(\tilde{A}_{n-a-1})}$ has a rational section. Thus $\tilde{G}^{-1}(\tilde{A}_{n-a-1})$ is a rational subvariety of codimension 2 of $\tilde{S}_{n,a}$. $\square$

For $a = 0$ the same method gives only that $\tilde{S}_n$ is covered by codimension 2 subvarieties which are stably rational of level 2.

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