COHERENT RINGS OF DIMENSION TWO

HO KUEN NG

ABSTRACT. We first continue the investigation of the finitely presented dimension, especially in the local case, then we propose a scheme for classifying commutative rings. Commutative coherent rings of finitely presented dimension two and global dimension two will be studied.

1. Introduction. Let R be a commutative ring with $1 \neq 0$ in this paper.

In [2], we introduced the finitely presented dimensions of modules and commutative rings. For A an R -module, we defined the finitely presented dimension of A , $\text{f.p.dim } A$, as $\inf\{n \mid \text{there exists an exact sequence } P_{n+1} \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0 \text{ of } R\text{-modules, where each } P_i \text{ is projective, and } P_{n+1}, P_n \text{ are finitely generated}\}$. If no such exact sequence exists, we say that A has infinite finitely presented dimension. We also defined the finitely presented dimension of the ring R , $\text{f.p.dim } R$, as $\sup\{\text{f.p.dim } A \mid A \text{ is a finitely generated } R\text{-module}\}$.

In [2], we showed that the finitely presented dimension has the properties that we expect of a 'dimension' if the ring in question is coherent. It was also proved that for a coherent ring R , $\text{gl.dim } R = \sup(\text{w.gl.dim } R, \text{f.p.dim } R - 1)$.

In §2 of this paper, we investigate the finitely presented dimension of a quasi-local ring, and obtain results concerning coherent rings of finitely presented dimension 2. We then point out a scheme for classifying rings according to their global dimensions, weak global dimensions and finitely presented dimensions. In §3, we show that the classification of quasi-local rings of global dimension 2 in [3] can be conveniently described by our scheme, and we look at the spectrum of prime ideals of coherent rings of global dimension 2.

2. Finitely presented dimension of coherent quasi-local rings. We first make the following

DEFINITION 2.1. Let R be a ring. An R -module A is said to be almost finitely presented if $A = B \oplus C$, where B is finitely presented, and C is nonfinitely generated free.

THEOREM 2.2. *Let R be a ring over which projective modules are free. Let A be an R -module with $\text{f.p.dim } A = 1$. Then A is almost finitely presented.*

PROOF. Let $0 \rightarrow K \rightarrow F \rightarrow A \rightarrow 0$ be an exact sequence with K finitely presented and F projective, and hence free. We fix a finite set of generators for K , and a set of free generators for F . We express the generators of K as unique linear combinations of the free generators of F . Let $f_1, \dots, f_n$ be involved in these linear combinations. Let $a_1, \dots, a_n$ be their images under the map $F \rightarrow A$, and let $B = (a_1, \dots, a_n)$. Let C be the submodule of A generated by the images of the other free generators of F , say $f_j \mapsto a_j$. Clearly we have $A = B + C$.

If $r_1 a_1 + \dots + r_n a_n = r_{n+1} a_{n+1} + \dots + r_{n+s} a_{n+s}$, where the r_i 's are in R , and $a_{n+1}, \dots, a_{n+s}$ are among the generators of C as described above, then $r_1 f_1 + \dots + r_n f_n - (r_{n+1} f_{n+1} + \dots + r_{n+s} f_{n+s})$ is in K , and hence is a linear combination of $f_1, \dots, f_n$. By the freeness of F , $r_{n+1} = \dots = r_{n+s} = 0$, showing that $B \cap C = 0$, and so $A = B \oplus C$.

The exact sequence $0 \rightarrow K \rightarrow (f_1, \dots, f_n) \rightarrow B \rightarrow 0$ shows that B is finitely presented. As $\text{f.p.dim } A = 1$, A is nonfinitely generated, and so F is nonfinitely generated. C is nonfinitely generated free because it is isomorphic to the submodule of F generated by the free generators other than $f_1, \dots, f_n$. This completes the proof.

PROPOSITION 2.3. *Let A be an R -module.*

- (1) $\text{f.p.dim } A = 0$ if and only if A is finitely presented.
- (2) If R is quasi-local and $\text{f.p.dim } A = 1$, then A is almost finitely presented.
- (3) If R is coherent and A is almost finitely presented, then $\text{f.p.dim } A = 1$.
- (4) If R is quasi-local coherent, then $\text{f.p.dim } A = 1$ if and only if A is almost finitely presented.

PROOF. (1) follows from definition. (2) follows from 2.2. (3) follows from the fact that a finitely generated submodule of a free module over a coherent ring is finitely presented. (4) is an immediate consequence of (2) and (3).

PROPOSITION 2.4. *A coherent quasi-local ring cannot have finitely presented dimension 2.*

PROOF. Else, there is an ideal I with $\text{f.p.dim } I = 1$, from [2]. Then $I = B \oplus C$, where B is finitely presented, and C is nonfinitely generated free. But C , being a free ideal of the ring, must be principal, a contradiction.

Among the consequences of 2.4, we see that a non-Noetherian hereditary ring cannot be quasi-local. In other words, a quasi-local hereditary ring must be Noetherian. Also, it follows from 2.4 that no valuation rings can have finitely presented dimension 2, a fact proved in [2].

However, we remark that if R is not quasi-local, $\text{f.p.dim } A = 1$ need not imply that A is almost finitely presented, because [2] showed that there is a coherent ring R with $\text{f.p.dim } R = 2$, and so there is an ideal with finitely presented dimension 1.

PROPOSITION 2.5. *Let R be non-Noetherian, and over which projective modules are free. Then $\text{f.p.dim } R \geq \text{gl.dim } R + 1 - \sup\{\text{p.d. } A \mid A \text{ is a finitely presented } R\text{-module}\}$.*

PROOF. It follows easily from the fact that over such a ring, a module with finitely presented dimension 1 is almost finitely presented, and so its projective dimension is the same as that of its finitely presented direct summand.

In [2], we saw that a ring is Noetherian if and only if it has finitely presented dimension 0. We said that the finitely presented dimension is a measure of how far away a ring is from being Noetherian. The following results point further in this direction.

PROPOSITION 2.6. *If $\text{f.p.dim } R = 2$, and R_p is coherent for all prime ideals p of R , then R is non-Noetherian, all localizations of which with respect to prime ideals are Noetherian.*

PROOF. It follows from [2] that a ring cannot have finitely presented dimension 1.

COROLLARY 2.7. *A coherent ring with finitely presented dimension 2 is non-Noetherian, all localizations of which with respect to prime ideals are Noetherian.*

For convenience, we call a ring R an (a, b, c) -ring if $\text{w.gl.dim } R = a$, $\text{gl.dim } R = b$, and $\text{f.p.dim } R = c$.

Before we move on to the next set of results, we need the following

PROPOSITION 2.8. *Let R be a coherent ring. Consider the following three conditions.*

- (1) $\text{w.gl.dim } R < \text{gl.dim } R$.
- (2) $\text{f.p.dim } R > \sup\{\text{f.p.dim } R_m \mid m \text{ is a maximal ideal}\}$.
- (3) $\text{gl.dim } R > \sup\{\text{gl.dim } R_m \mid m \text{ is a maximal ideal}\}$.

Assuming any one of these, the remaining two are equivalent.

PROOF. This follows by applying Theorem 3.4 of [2], and distinguishing between the finite and infinite cases.

Coherent rings were defined as a generalization of Noetherian rings. However some important properties possessed by the latter are not shared by the former, notably the Hilbert Basis Theorem. If we define $\mathcal{C}$ to be the class of coherent rings with finitely presented dimension ≤ 2 , then $\mathcal{C}$ is much more similar to the class of Noetherian rings. Examples of rings of finitely presented dimension 2 are given in [2].

PROPOSITION 2.9. *Let $R \in \mathcal{C}$. Then $\text{gl.dim } R < \infty$ if and only if Krull dimension $R < \infty$ and R_p is regular, for any prime ideal p of R .*

PROOF. We need only consider $\text{f.p.dim } R = 2$. For necessity, we note that each localization of R with respect to a prime ideal is regular local and has its Krull dimension = its global dimension $\leq \text{gl.dim } R < \infty$. For sufficiency, by [2], R is either a $(0, 1, 2)$ -ring, or an $(n, n, 2)$ -ring, with $n \neq 0$. For the latter case, every R_p , where p is a prime ideal of R , has $\text{gl.dim } R_p = \text{Krull dimension } R_p \leq \text{Krull}$

dimension $R < \infty$. By 2.7 and 2.8, since $\text{f.p.dim } R = 2 > \sup\{\text{f.p.dim } R_m \mid m \text{ is a maximal ideal of } R\}$, and $\text{w.gl.dim } R = \text{gl.dim } R$, we have

$$\begin{aligned}\text{gl.dim } R &= \sup\{\text{gl.dim } R_m \mid m \text{ is a maximal ideal of } R\} \\ &= \sup\{\text{gl.dim } R_p \mid p \text{ is a prime ideal of } R\}.\end{aligned}$$

This shows that $\text{gl.dim } R < \infty$.

In fact, the above result is also true for any coherent (n, n, m) -ring that is locally Noetherian.

The following result is obvious.

PROPOSITION 2.10. *$\mathcal{C}$ is closed under localization.*

PROPOSITION 2.11. *If $R \in \mathcal{C}$, and $\text{w.gl.dim } R < \infty$, then $R[X_1, X_2, \dots, X_n]$ is coherent, for any positive integer n .*

PROOF. By [1], it suffices to show that $R_p[X_1, X_2, \dots, X_n]$ is coherent, for all prime ideals p of R . This follows from 2.7 and the Hilbert Basis Theorem.

3. Rings of global dimension 2. In [3], it was shown that quasi-local rings of global dimension 2 are of three types: Noetherian rings, valuation rings and umbrella rings.

Since a quasi-local ring of global dimension 2 must be a coherent domain, and a domain with weak global dimension 0 must be a field, we conclude that a quasi-local ring of global dimension 2 must be a $(1, 2, 3)$ -, a $(2, 2, 0)$ -, or a $(2, 2, 3)$ -ring. It is clear that a $(1, 2, 3)$ -ring is a valuation ring, and that a $(2, 2, 0)$ -ring is a Noetherian ring. Hence a $(2, 2, 3)$ -ring must correspond to an umbrella ring.

Now let R be a coherent domain with global dimension 2. Then it must be a $(1, 2, 3)$ -, a $(2, 2, 0)$ -, a $(2, 2, 2)$ - or a $(2, 2, 3)$ -ring. We look at each of these rings.

If R is a $(1, 2, 3)$ -ring, then it is a Prüfer domain. We divide into two cases. In case one, $\text{gl.dim } R_p \leq 1$, for all prime ideals p of R . Then each R_p is a $(1, 1, 0)$ -ring, i.e. a discrete valuation ring, for any prime ideal $p \neq (0)$. Consequently, the Krull dimension of such a ring must be 1. In case two, there is a prime ideal p such that $\text{gl.dim } R_p = 2$. Such an R_p must be a $(1, 2, 3)$ -ring, and so cannot be a Noetherian ring, and hence not a discrete valuation ring.

A $(2, 2, 0)$ -ring is a Noetherian ring of Krull dimension 2. The spectrum is made up of three levels, the height 2 primes, the height 1 primes and the zero ideal.

A $(2, 2, 2)$ -ring is locally Noetherian. This shows that the Krull dimension of R is 2. If p has height 2, then pR_p contains at least two noncomparable prime ideals of R_p . Thus p contains at least two noncomparable prime ideals of R . By the Δ -Theorem in [3], p is finitely generated. By Cohen's theorem, there must be a prime ideal that is nonfinitely generated. Consequently, such a prime ideal must have height 1.

The spectrum of such a ring looks like that of a $(2, 2, 0)$ -ring, the difference being that some prime ideals in the middle level are nonfinitely generated.

The picture is more complicated for a $(2, 2, 3)$ -ring. Its localization with respect to a maximal ideal (or a nonzero prime ideal) is a $(1, 1, 0)$ -, a $(2, 2, 0)$ -, a $(1, 2, 3)$ -, or a $(2, 2, 3)$ -ring, i.e. a Noetherian ring of Krull dimension 1 or 2, a valuation ring of global dimension 2, or an umbrella ring. Hence its spectrum looks like the spectra of these rings laid side by side.

REFERENCES

1. B. Alfonsi, *Grade non-noethérien*, Comm. Algebra **9** (1981), 811–840.
2. H. K. Ng, *Finitely presented dimension of commutative rings and modules*, Pacific J. Math. (to appear).
3. W. V. Vasconcelos, *The rings of dimension two*, Dekker, New York, 1976.

DEPARTMENT OF MATHEMATICS, ALBION COLLEGE, ALBION, MICHIGAN 49224