EXAMPLES OF CENTRALIZERS OF AUTOMORPHISM GROUPS

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ABSTRACT. If AG is a finite solvable group with $(|A|, |G|) = 1$ and $G \triangleleft AG$, we explore the relationship between the Fitting height of G , the composition length of A and the structure of $C_G(A)$. For any A , we give examples where the difference between the Fitting height of G and that of $C_G(A)$ is exactly twice the composition length of A .

All groups in this paper are assumed to be finite. Let A be a solvable group and assume that A acts as automorphisms of a solvable group G such that $(|A|, |G|) = 1$. Then it is well known that the structure of $C_G(A)$ is intimately related to that of G . For example, it is known [3] that in that case, if h denotes Fitting height and k composition length, then $h(G) \leq 2k(A) + h(C_G(A))$. The purpose of this paper is to analyze to what extent this and related results are the best possible. We prove

THEOREM A. *Let A be any solvable group and h any integer greater than zero. Assume further that p and q are distinct primes not dividing $|A|$. Then there exists a (solvable) $\{p, q\}$ -group G and an action of A on G as automorphisms such that $h(C_G(A)) = h$ and $h(G) = 2k(A) + h$.*

The special case of Theorem A where $k(A) = 1$ is due to E. C. Dade and appears in [2]. We also prove the following, where F denotes Fitting subgroup.

THEOREM B. *Given any two solvable groups A and C such that $(|A|, |C|) = 1$, and any prime $p \nmid |A| |F(C)|$, there exists a solvable group G and an action of A on G by automorphisms such that $(|A|, |G|) = 1$, $F(C_G(A)) = O_p(C_G(A))$, $C_G(A)/O_p(C_G(A)) \simeq C$ and $h(G) = 2k(A) + h(C_G(A))$.*

If $A \neq 1$ and $C_G(A)$ has all its Fitting factors Abelian, it follows from [3, Theorem 3.1] that $h(G) \leq 2k(A) + h(C_G(A)) - 1$. One cannot therefore prescribe $C_G(A)$ arbitrarily if we want to have $h(G) = 2k(A) + h(C_G(A))$. Furthermore, we have by [3, Remark 3.3], that $h(G) \leq k(A) + h(C_G(A))$ if all Fitting factors of G are pairwise relatively prime. The special case of the next theorem where $C = 1$ is due to Gross [1].

THEOREM C. *Let A and C be solvable groups and assume $(|A|, |C|) = 1$. Then there exists a solvable group G and an action of A on G by automorphisms such that $(|A|, |G|) = 1$, $C_G(A) \simeq C$ and $h(G) = k(A) + h(C_G(A))$. Furthermore, if all ascending Fitting factors of C are pairwise of relatively prime order, G can be chosen with the same property.*

Given a solvable group C and a positive integer a define

$$m(a, C) = (1/a) \max\{h(G) - h(C_G(A))\}$$

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where A and G range over all solvable groups such that $k(A) = a$, $(|A|, |G|) = 1$, A acts on G by automorphisms and $C_G(A) \simeq C$. We know that $1 \leq m(a, C) \leq 2$, the first inequality by Theorem C and the second by [3, Corollary 3.2]. It is a well-known conjecture that $\sup_{a \geq 1} (m(a, 1)) = 1$. It follows from Theorem A that for infinitely many choices of C , $\sup_{a \geq 1} (m(a, C)) = 2$. Nevertheless, the following seems plausible.

CONJECTURE. For any solvable C , $\lim_{a \rightarrow \infty} (m(a, C)) = 1$.

2. Proofs. In this section we construct the relevant examples.

LEMMA 2.1. *Let B be a subgroup of a group A . Assume that B acts on a group G . Then there exists a group G^* isomorphic to the direct product of $|A : B|$ copies of G and an action of A on G^* such that $C_{G^*}(A) \simeq C_G(B)$, and if $B \triangleleft A$ then $C_{G^*}(B)$ is the direct product of $|A : B|$ copies of $C_G(B)$.*

PROOF. This, of course, is just induction. For example, we may imbed A in the usual way by a Frobenius map (see for example [4, Proposition 1.6]) into the wreath product $S_n \sim B$, where $n = |A : B|$ and S_n is the symmetric group. This provides a natural action of A on G^* . If $x \in C_{G^*}(A)$ then it is centralized by $B \times 1 \times \cdots \times 1$. By taking conjugates we get that x is centralized by the normal subgroup $B^{|A:B|}$ of $S_n \sim B$. But then A acts on x just by transitively permuting the indices, so all the coordinates of x are equal and are centralized by B . Hence $C_{G^*}(A) \simeq C_G(B)$ as desired. If $B \triangleleft A$ then B is mapped into $B^{|A:B|}$ by the Frobenius map and $C_{G^*}(B) = [C_G(B)]^{|A:B|}$. This proves the lemma.

LEMMA 2.2. *Assume that A is a group and $B \triangleleft A$. Assume further that A acts on two groups G_1 and G_2 of order g_1 and g_2 , respectively, and that B centralizes G_1 . Let $G_1 \sim G_2$ denote the wreath product of order $g_1 g_2^{g_1}$. Then A acts in a natural way on $G_1 \sim G_2$, the action of A on the natural image of G_1 in $G_1 \sim G_2$ is equivalent to that of A on G_1 and $C_{G_2 g_1}(B) = [C_{G_2}(B)]^{g_1}$.*

PROOF. $G_1 \sim G_2$ is the set of ordered pairs (π, f) where π is a permutation of G_1 obtained by left multiplication by some fixed element of G_1 and f is a function from G_1 to G_2 .

$$(\pi, f)(\pi', f') = (\pi \circ \pi', (f \circ \pi') \cdot f')$$

where $\circ$ denotes composition of functions, and $\cdot$ product by components. We think of the elements of A as permutations of G_1 and G_2 , and define, for $a \in A$,

$$a((\pi, f)) = (a \circ \pi \circ a^{-1}, a \circ f \circ a^{-1}).$$

For any $g \in G_1$ let π be its corresponding Cayley permutation and set $\sigma(g) = (\pi, 1)$, where $1(x) = 1$ for all $x \in G_1$. Then

$$a(\sigma(g)) = (a \circ \pi \circ a^{-1}, a \circ 1 \circ a^{-1}) = (a \circ \pi \circ a^{-1}, 1)$$

and if $x \in G_1$, $(a \circ \pi \circ a^{-1})(x) = a(g)x$. So the action of A on the natural image of G_1 in $G_1 \sim G_2$ is equivalent to the action of A on G_1 .

Since a provides an automorphism of G_2 it is easy to verify that a provides an automorphism of $G_1 \sim G_2$ and that this gives a representation of A as automorphisms of $G_1 \sim G_2$.

If $b \in B$ then b acts as the trivial permutation on G_1 and it acts on (Id, f) by $b((\text{Id}, f)) = (\text{Id}, b \circ f)$. Whence we get $C_{G_2} g_1(B) = [C_{G_1}(B)]^{g_1}$. This concludes the proof of 2.2.

We define as usual $F_0(G) = 1$ and $F_i(G)/F_{i-1}(G) = F(G/F_{i-1}(G))$ for all positive integers i .

LEMMA 2.3. *Let A be any solvable group. Let $h = k(A)$ and let p_i , $i = 1, \dots, h$, be a finite sequence of primes. Let p be a prime such that $p \nmid p_1 \cdots p_h |A|$, and assume $(p_1 \cdots p_h |A|) = 1$. Then there exists a solvable group G and an action of A on G by automorphisms such that:*

1. $\pi(G) = \{p, p_i: i = 1, \dots, h\}$.
2. $h(G) = 2h$.
3. $F_i(G)/F_{i-1}(G)$ is an elementary Abelian p_j -group if $i = 2j$ and an elementary Abelian p -group if i is odd.
4. If $p_i \neq p_{i+1}$, $i = 1, \dots, h-1$, and H is a Hall p' -subgroup of G then $h(H) = h$ and $F_i(H)/F_{i-1}(H)$ is an elementary Abelian p_i -group, $i = 1, \dots, h$.
5. $C_G(A)$ is a p -group.

PROOF. Assume false. Let A be a counterexample with $k(A)$ minimal. Assume first that $k(A) > 1$. Let then B be a normal subgroup of A such that $k(B) + 1 = k(A)$. By induction we may find G_1 satisfying the conditions for $p_1, \dots, p_{h-1}$ and B , and G_2 satisfying them for p_h and A/B . By Lemma 2.1 we may assume A acts on G_1 and, by Lemma 2.2, A acts on the wreath product $W = G_2 \sim G_1$, and it is easy to verify that W satisfies properties 1–5. So since the result is clear if $k(A) = 0$, we have $k(A) = 1$. Let A act by multiplication on the additive group F of the extension of $\mathbb{F}_{p_1}$ by a primitive $|A|$ th root of 1. Let M be a faithful $\mathbb{F}_p A F$ -module. Then set $G = FM$. It is clear that G satisfies 1–5. This contradiction concludes the proof of Lemma 2.3.

PROOF OF THEOREM A. Let D be a solvable $\{p, q\}$ -group such that $h(D) = h$ and $F(D) = O_p(D)$. Let G satisfy the condition of Lemma 2.3 with the sequence $p_i = q$, $i = 1, \dots, k(A)$. Then $W = D \sim G$ satisfies the conditions of Theorem A by Lemma 2.2, since $C_W(A)$ is an extension of D by a p -group and $h(C_W(A)) = h$.

PROOF OF THEOREM B. Let $D = C \sim \mathbf{Z}_p$ and choose a prime q such that $q \nmid p|A|$. Use the construction in the proof of Theorem A.

PROOF OF THEOREM C. Take $p_1, \dots, p_h$, $h = k(A)$ a sequence of primes such that $p_i \neq p_{i+1}$, $i = 1, \dots, h = 1$, $(p_1 \cdots p_h, |A|) = 1$ and $p_h \nmid |F(C)|$. Let H be an A -invariant Hall p' -subgroup of the group G given by Lemma 2.3. Take $C \sim H$. By Lemma 2.2 we get the theorem.

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