CONVOLUTION INEQUALITIES ON FINITE CYCLIC GROUPS AND THE PSEUDOMEASURE NORM

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ABSTRACT. A characterization of the probability measures that define hypercontractive convolution operators on finite cyclic groups has been given in terms of the pseudomeasure norm. Here the pseudomeasure norm is shown to be a poor quantitative predictor of hypercontractiveness in an asymptotic sense.

For $K \geq 2$, let G_K denote the additive cyclic group $Z/KZ = \{0, 1, \dots, K-1\}$, and for $p \geq 1$, let $L^p(G_K)$ denote the usual Banach space of p th power summable functions on G_K with norm $\|\cdot\|_{p,K}$ defined in terms of the Haar measure m_K that assigns mass $1/K$ to each point. Let Γ_K denote the group dual to G_K supplied with ordinary counting measure so that the Plancherel theorem is valid for the pair of groups G_K, Γ_K . Finally, let $P(G_K)$ denote the set of probability measures on G_K .

Fix $p > 1$ and μ in $P(G_K)$. If there is a $q > p$ such that

$$(1) \quad \|\mu * f\|_{q,K} \leq \|f\|_{p,K}, \quad f \in L^p(G_K),$$

then we say μ is a *hypercontraction*.

The interest in hypercontractive measures on finite cyclic groups originated in the study of convolution properties of particular measures defined on the circle group, the Cantor group, and the real line; see [1–5, and 8]. Currently, however, the dependence of p and q in (1) on the measure μ is really completely understood on only one group: $G_2 = Z/2Z$. In this case, Bonami [3], Beckner [1], and Weissler [8] have shown that

$$(2) \quad (q-1)|\hat{\mu}(1)|^2 \leq p-1$$

is necessary and sufficient for (1) to be true, where $\hat{\mu}$ is the Fourier-Stieltjes transform of μ .

Although the determination of the precise relationship between p and q in (1) for an arbitrary measure μ is difficult, a characterization of hypercontractive probability measures in terms of the pseudomeasure norm is known [2, 6]:

THEOREM 1. (a) μ in $P(G_K)$ satisfies (1) for some $q > p$ if, and only if,

$$(3) \quad \|(m_K - \mu)^\wedge\|_\infty < 1.$$

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(b) In addition, if $p > 1$ and $0 < \xi < 1$, then there is a $q > p$ such that (1) is true for μ in $P(G_K)$ whenever

$$(4) \quad \|(m_K - \mu)^\wedge\|_\infty \leq \xi.$$

If we let $q(K, \xi, p)$ denote the supremum of the $q > p$ such that (4) implies (1) on G_K , then our main result is the theorem that follows.

THEOREM 2. *If $p > 1$ and $0 < \xi < 1$, then $\lim_{K \rightarrow \infty} q(K, \xi, p) = p$.*

To prove Theorem 2, we work on the circle group $T = R/Z$ which we realize as the interval $[0, 1)$. Consequently, for $K \geq 2$, we take $G_K = \{j/K: j = 0, 1, \dots, K-1\}$. Then it turns out that our theorem is a consequence of the following two lemmas.

LEMMA 1. *Let μ be a Borel measure on the circle group, and let $p \geq 1$. If there is a $q > p$ and $C > 0$ such that*

$$(5) \quad \|\mu * f\|_q \leq C \|f\|_p$$

for each f in $L^p(T)$, then μ is a continuous measure.

LEMMA 2. *For each $K \geq 2$, let μ_K be a probability measure on T with support in G_K . Suppose there is a $p \geq 1$ and $q > p$ such that (1) is true for μ_K for infinitely many K . If μ is a Borel measure on T such that $\mu_K \rightarrow \mu$ weak* as $K \rightarrow \infty$, then μ satisfies (5) for f in $L^p(T)$ with $C = 1$.*

PROOF OF LEMMA 1. By a standard interpolation involving Young's inequality for measures and the Riesz-Thorin interpolation theorem, we may suppose $q = 2$ and $p < q$ in (5). By convolving μ with the Dirichlet kernel $D_K(x) = \sum_{|j| \leq K} \text{Exp}(2\pi i j x)$, using the fact that $\|D_K\|_p \sim K^{1/p'}$, where $(1/p) + (1/p') = 1$, and applying (5), we obtain

$$\sum_{|j| \leq K} |\hat{\mu}(j)|^2 \leq C_0 K^{2/p'} \leq C_0 (2K+1)^{2/p'},$$

where C_0 is independent of K . Since $p' > 2$, the lemma follows from Wiener's theorem on the average value of $|\hat{\mu}|^2$ once we divide both sides of the inequality above by $2K+1$ and let $K \rightarrow \infty$. $\square$

PROOF OF LEMMA 2. It suffices to show (5) holds with f a continuous complex-valued function on the circle. Thus, let f be continuous, and for $K \geq 2$, let f_K be the restriction of f to G_K , and let $I(K, j) = [j/K, (j+1)/K)$ for $j = 0, \dots, K-1$. The sequence of functions $\{\mu_K * f\}$ is uniformly equicontinuous, and since $\mu_K \rightarrow \mu$ weak*, we have $\mu_K * f \rightarrow \mu * f$ pointwise. Consequently,

$$(6) \quad \|\mu_K * f - \mu * f\|_\infty \rightarrow 0 \quad \text{as } K \rightarrow \infty.$$

Next, from

$$\|\mu_K * f\|_q^q - \|\mu_K * f_K\|_{q,K}^q = \sum_{j=0}^{K-1} \int_{I(K,j)} |\mu_K * f(x)|^q - |\mu_K * f_K(j/K)|^q dx$$

and

$$\mu_K * f(x) - \mu_K * f_K\left(\frac{j}{K}\right) = \sum_{i=0}^{K-1} \left[f\left(x - \frac{i}{K}\right) - f\left(\left(\frac{j-i}{K}\right)\right) \right] \mu_K\left(\frac{i}{K}\right)$$

it is easy to see

$$(7) \quad \left| \|\mu_K * f\|_q - \|\mu_K * f_K\|_{q,K} \right| \rightarrow 0 \quad \text{as } K \rightarrow \infty.$$

Similarly, we have

$$(8) \quad \left| \|f\|_p - \|f_K\|_{p,K} \right| \rightarrow 0 \quad \text{as } K \rightarrow \infty.$$

To see how (5) follows from (1) and (6)–(8), let $\varepsilon > 0$. Then for some K sufficiently large we have

$$\begin{aligned} \|\mu * f\|_q &\leq \|\mu * f - \mu_K * f\|_q + \|\mu_K * f\|_q \\ &\leq \|\mu * f - \mu_K * f\|_\infty + \|\mu_K * f_K\|_{q,K} + \varepsilon \\ &\leq \|f_K\|_{p,K} + 2\varepsilon \leq \|f\|_p + 3\varepsilon, \end{aligned}$$

and the lemma is proved. $\square$

We now prove Theorem 2.

PROOF OF THEOREM 2. Fix $p > 1$ and $0 < \xi < 1$. By Theorem 1(b), $q(K, \xi, p) > p$ for each $K \geq 2$. Therefore, it suffices to show

$$(9) \quad \limsup q(K, \xi, p) = p.$$

We obtain (9) by an indirect argument. Suppose (9) is false, and fix q with

$$\limsup q(K, \xi, p) > q > p.$$

Then for infinitely many K , on G_K we have (4) implying (1) with ξ , p , and q as above. For each $K \geq 2$, define μ_K on G_K by $\mu_K = \xi\delta_0 + (1 - \xi)m_K$, where δ_0 is the point mass at 0 on G_K . Then for j in Γ_K , $(m_K - \mu_K)(j) = -\xi$ when $j \neq 0$, and $(m_K - \mu_K)(0) = 0$. Therefore μ_K satisfies (4). Consequently (1) is true for infinitely many μ_K . From Lemma 2, $\mu = \xi\delta_0 + (1 - \xi)dx$, the weak* limit of the μ_K 's, satisfies (5), and hence by Lemma 1 is continuous, a patent absurdity. $\square$

REMARKS. (a) The method of proof of Theorem 1 found in [2] is superior to that of [6] because the method used in [2] provides an estimate of $q(K, \xi, p)$. By using the elementary inequality $\|f\|_{4,K} \leq K^{1/4}\|f\|_{2,K}$ and by following the line of the proof of Lemma 2 of [2], one can readily show

$$q(K, \xi, 2) \geq \frac{4 \ln(\xi^{-1}) - 2 \ln(3K^{1/2})}{\ln(\xi^{-1}) - \ln(3K^{1/2})}.$$

Consequently, an application of the Riesz-Thorin interpolation theorem and Young's inequality for measures yields a lower bound for $q(K, \xi, p)$ whenever $1 < p < \infty$.

(b) Each measure μ_K appearing in the proof of Theorem 2 is obviously extremal with respect to p and ξ satisfying $q(K, \xi, p) = 2$. It would be of interest to know whether it is extremal for every p and ξ .

(c) An earlier version of this paper claimed, erroneously, to have settled a conjecture due to Beckner, Janson, and Jerison found in [2]. To our knowledge that conjecture remains open.

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