CLEAR VISIBILITY AND SETS WHICH ARE ALMOST STARSHAPED

MARILYN BREEN

ABSTRACT. Let S be a nonempty compact connected set in R^2 . Assume that every two points of local nonconvexity of S are clearly visible from a common point of S . Then S is "almost" starshaped. In fact, for any point y in R^2 there is a line L through y such that $L \cap S$ is convex and every point of S sees via S a point of L . If "clearly visible" is replaced by the weaker term "visible", then the result fails.

1. Introduction. We begin with some preliminary definitions. Let S be a subset of the plane. A point s in S is said to be a *point of local convexity* of S if and only if there is some neighborhood N of s such that $N \cap S$ is convex. If S fails to be locally convex at point q in S , then q is called a *point of local nonconvexity* (lnc point) of S . For points x and y in S , we say x *sees* y via S (x is *visible* from y via S) if and only if the corresponding segment $[x, y]$ lies in S . Point x is *clearly visible* from y via S if and only if there is some neighborhood N of x such that y sees each point of $S \cap N$ via S . Set S is an L_2 set if and only if every two points of S are visible via S from a common point. Finally, S is *starshaped* if and only if there is some point p in S such that p sees each point of S via S , and the set of all such points p is called the (convex) *kernel* of S , denoted $\ker S$.

A well-known theorem of Krasnosel'skiĭ [4] states that if S is a nonempty compact set in R^2 , then S is starshaped if and only if every 3 points of S are visible via S from a common point. If the number 3 is reduced to 2, then the result fails, for a compact L_2 set need not be even a finite union of starshaped sets [2, Example 1]. However, this leads to a related question examined in [2]: When will a closed planar set be "almost" starshaped? That is, when will there exist a convex subset C of S such that every point of S sees via S some point of C ?

In several studies of starshaped sets, the notion of clearly visible has been a useful tool. Here we use this concept to obtain the following result: Let S be a nonempty compact connected set in R^2 . Assume that every 2 points of local nonconvexity of S are clearly visible from a common point of S . Then S is almost starshaped. In fact, for any point y in R^2 there exists a line L through y such that $L \cap S$ is convex and every point of S sees via S a point of L . If "clearly visible" is replaced by the weaker term "visible", then the result fails.

The following familiar terminology will be used throughout the paper: $\text{conv } S$ and $\text{cl } S$ will denote the convex hull and closure, respectively, for set S , and $\text{lnc } S$ will be the set of points of local nonconvexity of S . For distinct points x and y , $L(x, y)$ will be the line determined by x and y , and $R(x, y)$ will be the ray from

Received by the editors August 5, 1983.

1980 *Mathematics Subject Classification*. Primary 52A30, 52A35.

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 0002-9939/84 \$1.00 + \$.25 per page

x emanating through y . Finally, $\text{cone}(x, A)$ will represent $\bigcup\{R(x, a) : a \in A, a \neq x\}$. The reader is referred to Valentine [7] and Lay [5] for a discussion of these concepts.

2. The results. We will be concerned with the proof of the following theorem.

THEOREM 1. *Let S be a nonempty compact connected set in R^2 . Assume that every 2 points of local nonconvexity of S are clearly visible from a common point. Then S is "almost" starshaped. In fact, for any point y in R^2 there is a line L through y such that $L \cap S$ is convex and every point of S sees via S some point of L .*

PROOF. If S is convex, the result is immediate. Thus we may assume that S is not convex and hence has lnc points by a theorem of Tietze [6]. For each lnc point q of S , define the set $A_q = \{x : q \text{ is clearly visible from } x \text{ via } S\}$. Using our hypothesis, every 2 of the compact convex sets in $\mathcal{A} \equiv \{\text{conv cl } A_q : q \text{ in lnc } S\}$ intersect. Therefore, by a result of Horn [3, Theorem 4], for any point y in R^2 there is a line L through y which intersects every member of $\mathcal{A}$. We will show that L satisfies the theorem.

The proof will be accomplished by a sequence of assertions.

ASSERTION 1. *$S \cap L$ is convex:* Suppose on the contrary that there exist points $a, b \in S \cap L$ with $(a, b) \cap S = \emptyset$, to obtain a contradiction. By [8, Lemma 1], each point of S sees via S some lnc point, so a sees an lnc point, as does b . Moreover, by hypothesis, every 2 lnc points are visible from a common point of S . Hence there is a polygonal path λ in S from a to b , where λ has length at most 4 and λ has no multiple points. Choose $a', b' \in \lambda \cap L$ such that $a' \leq a < b \leq b'$ and such that there are no points of L on the subpath λ' of λ joining a' to b' (except for a' and b' , of course).

Consider the closed simply connected subset T of R^2 bounded by $\lambda' \cup [a', b']$. Let K be the component of $T \sim S$ which contains (a, b) . Let L' be a line parallel to L and distinct from L supporting $\text{conv cl } K$, and let q be an extreme point of $\text{conv cl } K$ on L' . Using an argument similar to the one in [1, Lemma 3, Case 1], it is not hard to show that q is an lnc point of S and q is clearly visible via S only from points in the closed halfspace $\text{cl } L'_1$ determined by L' and disjoint from L . However, this implies that $\text{conv cl } A_q \subseteq \text{cl } L'_1 \subseteq \sim L$, impossible since $(\text{conv cl } A_q) \cap L \neq \emptyset$. We have a contradiction, and $L \cap S$ must be convex.

ASSERTION 2. *For $q \in \text{lnc } S$, q sees via S some point of $L \cap \text{cone}(q, \text{cl } A_q)$:* By our choice of L , it is easy to see that $\text{cl } A_q$ cannot lie in either of the open halfplanes L_1 or L_2 determined by L . Hence we may select points $p \in \text{cl } A_q \cap \text{cl } L_1$ and $r \in \text{cl } A_q \cap \text{cl } L_2$. Without loss of generality, assume that $q \in \text{cl } L_1$. Then $[q, r]$ meets L at some point s , and s satisfies the assertion.

ASSERTION 3. *Set S is simply connected:* Suppose on the contrary that $\sim S$ contains a bounded component B . Let M and N be distinct parallel lines supporting $\text{conv cl } B$, and let c and d be extreme points of $\text{conv cl } B$ in M and in N , respectively. Using an argument from [1, Lemma 3, Case 1], c and d are lnc points of S . Moreover, if the corresponding halfplanes are labeled so that $\text{cl } M_1 \cap \text{cl } N_2 = \emptyset$, then $A_c \subseteq \text{cl } M_1$, $A_d \subseteq \text{cl } N_2$, and c and d cannot be clearly visible from a common point of S , contradicting our hypothesis.

ASSERTION 4. *For x in S , x sees via S some point of line L :* If x is an lnc point, the argument is finished by Assertion 2 above, so assume that x is not an

FIGURE 1

FIGURE 2

lnc point. By [8, Lemma 1], x sees via S some lnc point q . Moreover, by Assertion 2, q sees via S some point $p \in L \cap \text{cone}(q, \text{cl } A_q)$. Now if $[x, q]$ meets L or if x, q, p are collinear, we are finished. Otherwise, choose point w on $[q, p]$ closest to p such that $\text{conv}\{x, q, w\} \subseteq S$. If $w = p$, again we are through, so assume $w \neq p$. Then by an argument involving [8, Corollary 1], $[x, w]$ contains an lnc point q_2 of $\text{conv}\{x, q, p\} \cap S$. Clearly $q_2 \in \text{lnc } S$ and since $x \notin \text{lnc } S$, $q_2 \neq x$. Moreover, it is not hard to see that A_{q_2} lies in the closed halfplane $\text{cl } M_2$ determined by $M \equiv L(x, w)$ and disjoint from p . Again using Assertion 2, q_2 sees via S some point p_2 of $L \cap \text{cone}(q_2, \text{cl } A_{q_2})$, and since $A_{q_2} \subseteq \text{cl } M_2$, $p_2 \in \text{cl } M_2$. Observe that L and $L(x, w)$ cannot be parallel, and let r denote their point of intersection. Then $p_2 \leq r < p$ on L . Since $L \cap S$ is convex, $[p_2, p] \subseteq S$. Hence

$$[x, q] \cup [q, w] \cup [w, q_2] \cup [q_2, p_2] \cup [p_2, p] \cup [p, w] \cup [w, x] \subseteq S.$$

Using the fact that S is simply connected, it is easy to see that $[x, r] \subseteq S$, and the assertion is established.

This completes the proof of the theorem.

It is interesting to note that the result in Theorem 1 fails if “clearly visible” is replaced by the weaker term “visible”, as the following example illustrates.

EXAMPLE 1. Let S be the set in Figure 1. Every 2 (in fact, all) lnc points of S are visible from a common point, yet S is not almost starshaped.

Finally, the condition in Theorem 1 is sufficient but not necessary for a set to be almost starshaped, as Example 2 reveals.

EXAMPLE 2. Let S be the set in Figure 2. The 2 lnc points q and r of S are clearly visible from no common point, yet every point of S sees via S some point of segment $[q, r]$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OKLAHOMA, NORMAN, OKLAHOMA
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