A COMPARISON THEOREM FOR SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS

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ABSTRACT. A new kind of comparison theorem in which an SDE is compared with two deterministic ODEs is established by means of the generalized sample solutions of SDEs. Using this theorem, we can compare solutions of two SDEs with different diffusion coefficients and obtain some asymptotic estimations for the paths of diffusion processes.

In the investigation of solutions of stochastic differential equations (SDEs), comparison theorems are very powerful tools as in the case for deterministic ones. But so far most of these theorems have dealt with those SDEs with the same diffusion coefficient (cf. [1, 3–7, 9, 10]) except in [8] where a very special case involving two different diffusion coefficients has been discussed (cf. Example 2).

In this paper, we use the method of generalized sample solutions of SDEs (cf. [5, 11]) to establish a new kind of comparison theorem in which an SDE is compared with two deterministic ODEs. The conditions imposed here are weaker than those in [5 and 6] and the proof is much simpler. On the other hand, a discontinuous right-hand side is allowed. So it seems more appropriate to the stochastic optimal control problems.

We begin with a lemma.

LEMMA. Assume that two functions $f(t, x)$ and $\tilde{f}(t, x)$ are defined on some domain G in $\mathbf{R}^2$ satisfying the Carathéodory conditions, that is, they are measurable in t , continuous in x and dominated by a locally integrable function $m(t)$ in the domain G . Let (t_0, x_0) and $(t_0, \tilde{x}_0)$ be two points in G such that $x_0 \leq \tilde{x}_0$, $x(t)$ be any solution to the initial value problem

$$(1) \quad \dot{x}_t = f(t, x_t), \quad x(t_0) = x_0$$

and $\tilde{x}(t)$ be the maximal solution to the problem

$$(2) \quad \dot{x}_t = \tilde{f}(t, x_t), \quad x(t_0) = \tilde{x}_0.$$

If the inequality

$$(3) \quad (t - t_0)f(t, x) \leq (t - t_0)\tilde{f}(t, x)$$

holds a.e. in G , then $x(t) \leq \tilde{x}(t)$ for every t in the common interval of existence of the solutions $x(t)$ and $\tilde{x}(t)$.

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PROOF. Note that condition (3) means that if $t > t_0$, then $f \leq \tilde{f}$, and if $t < t_0$, then $f \geq \tilde{f}$. In the case that $t > t_0$, the lemma is an immediate consequence of Theorem 7 in [2]. The other half can be proved in a similar way.

REMARK. If the condition (3) is replaced by

$$(3') \quad (t - t_0)f(t, x) \geq (t - t_0)\tilde{f}(t, x) \quad \text{a.e. in } G,$$

if $x_0 \geq \tilde{x}_0$ and if $\tilde{x}(t)$ is the minimal solution to problem (2), then $x(t) \geq \tilde{x}(t)$ holds for every t in the common interval of existence.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $(\mathcal{F}_t, t \geq t_0)$ an increasing family of sub- σ -algebras of $\mathcal{F}$ satisfying the usual conditions. Now we consider the following SDE:

$$(4) \quad \begin{cases} dX_t = b(t, W_t, X_t) dt + \sigma(t, W_t, X_t) dW_t, & t \geq t_0, \\ X(t_0) = x_0, \end{cases}$$

where $\{W_t, t \geq t_0\}$ is one-dimensional Brownian motion, x_0 is an $\mathcal{F}_{t_0}$ -measurable random variable and b and σ are two functions defined and continuous in some domain G in $[t_0, \infty) \times \mathbf{R}^2$.

Let us first consider the following deterministic initial value problem:

$$(5) \quad d\Phi/dw = \sigma(t, w, \Phi), \quad \Phi(w_0) = \xi$$

where t serves as a parameter. If, moreover, $\sigma \in C^2(G)$, then the problem (5) admits a unique C^2 -solution $\Phi(w, t, \xi, w_0)$ in some domain $\tilde{G}$.

Define

$$(6) \quad g(t, \xi, \omega) \equiv \Phi_\xi^{-1}[b - \frac{1}{2}(\sigma_w + \sigma_x \sigma) - \Phi_t]$$

where the subscripts indicate the partial derivatives and the arguments w and x are replaced by $W(t, \omega)$ and $\Phi(W(t, \omega), t, \xi, w_0)$ respectively. If the function b satisfies a local Lipschitz condition with respect to x , then the following initial value problem

$$(7) \quad d\xi/dt = g(t, \xi, \omega), \quad \xi(t_0) = x_0(\omega)$$

admits a unique solution $\xi(t, \omega)$, $t_0 \leq t < \varsigma(\omega)$, where ω serves as a parameter and ς is the "explosion time".

The following theorem is a localized version of those theorems in [5 and 11].

THEOREM 1. *Assuming that $\sigma \in C^2(G)$ and that $b \in C^0(G)$ and locally Lipschitz continuous with respect to x in G , the SDE (4) has a unique strong solution*

$$(8) \quad X(t, \omega) = \Phi(W(t, \omega), t, \xi(t, \omega), w_0), \quad t_0 \leq t < \varsigma(\omega),$$

where Φ and ξ are the solutions to the problems (5) and (7), respectively, and ς is the "explosion time" for X_t .

PROOF. Apply the same procedure used in [11]; we can show that $\xi(t, \omega)$ is an adapted continuous process. Choose a sequence of compact sets $\{K_n\}$ in G such that $K_n \uparrow G$ and define

$$\varsigma_n(\omega) = \inf\{t > t_0: (t, W(t, \omega), \Phi(W(t, \omega), t, \xi(t, \omega), w_0)) \notin K_n\}.$$

It follows that each ς_n is a stopping time and that $\varsigma_n \uparrow \varsigma$. Applying the Itô formula to the process

$$X(t \wedge \varsigma) = \Phi(W(t \wedge \varsigma), t \wedge \varsigma, \xi(t \wedge \varsigma), w_0),$$

we have

$$\begin{aligned}
 X(t \wedge \zeta) - x_0 &= \int_{t_0}^{t \wedge \zeta} \Phi_w(W_s, s, \xi_s, w_0) dW_s \\
 &\quad + \int_{t_0}^{t \wedge \zeta} \left(\frac{1}{2} \Phi_{ww} + \Phi_t \right) ds + \int_{t_0}^{t \wedge \zeta} \Phi_\xi d\xi_s \\
 &= \int_{t_0}^{t \wedge \zeta} \sigma(s, W_s, X_s) dW_s + \int_{t_0}^{t \wedge \zeta} \left[\frac{1}{2} (\sigma_w + \sigma_x \sigma) + \Phi_t \right] ds \\
 &\quad + \int_{t_0}^{t \wedge \zeta} \Phi_\xi \cdot g(s, \xi_s, \omega) ds \\
 &= \int_{t_0}^{t \wedge \zeta} \sigma(s, W_s, X_s) dW_s + \int_{t_0}^{t \wedge \zeta} b(s, W_s, X_s) ds \quad \text{a.s.}
 \end{aligned}$$

which shows that $\{X_t, t \in [t_0, \zeta)\}$ is a solution to SDE (4).

Conversely, if $\{\tilde{X}_t, t \in [t_0, \tilde{\zeta})\}$ is a solution to SDE (4), we can also use the Itô formula to the process

$$\tilde{\xi}(t \wedge \tilde{\zeta}) \equiv \Phi(w_0, t \wedge \tilde{\zeta}, \tilde{X}(t \wedge \tilde{\zeta}), W(t \wedge \tilde{\zeta}))$$

to prove that $\{\tilde{\xi}(t), t \in [t_0, \tilde{\zeta})\}$ is a solution to problem (7) as it was done in [5 or 11]. It follows from the uniqueness of the solutions to problem (7) that $\tilde{\zeta} \leq \zeta$ and $\tilde{\xi}(t)$ coincides with $\xi(t)$ in the interval $[t_0, \tilde{\zeta})$. Therefore,

$$\begin{aligned}
 \tilde{X}(t \wedge \tilde{\zeta}) &= \Phi(W(t \wedge \tilde{\zeta}), t \wedge \tilde{\zeta}, \tilde{\xi}(t \wedge \tilde{\zeta}), w_0) \\
 &= \Phi(W(t \wedge \tilde{\zeta}), t \wedge \tilde{\zeta}, \xi(t \wedge \tilde{\zeta}), w_0) = X(t \wedge \tilde{\zeta}) \quad \text{a.s.}
 \end{aligned}$$

which establishes the pathwise uniqueness of the solutions to SDE (4) and the proof is complete.

Now we give a comparison theorem which is an immediate consequence of this theorem and the preceding lemma.

THEOREM 2. *Under the assumption of Theorem 1, if there exists a function $\tilde{\sigma}$ defined on G satisfying the Carathéodory conditions and the inequality*

$$(9) \quad (w - w_0)\sigma(t, w, x) \leq (w - w_0)\tilde{\sigma}(t, w, x) \quad \text{a.e. in } G,$$

and if, moreover, there exists a function $\tilde{g}$ defined on D , the domain of g , satisfying the Carathéodory conditions and the inequality

$$(10) \quad g(t, \xi, \omega) \leq \tilde{g}(t, \xi, \omega) \quad \text{in } D,$$

then for the unique solution X_t of SDE (4)

$$(11) \quad X(t) \leq \tilde{\Phi}(W(t), t, \tilde{\xi}(t), w_0)$$

holds for a.e. ω and every t in the common interval where both sides are defined. Here $\tilde{\Phi}$ and $\tilde{\xi}$ are the maximal solutions to the problems

$$(5) \quad \frac{d\Phi}{dw} = \tilde{\sigma}(t, w, \Phi), \quad \Phi(w_0) = \xi$$

and

$$(7) \quad \frac{d\xi}{dt} = \tilde{g}(t, \xi, \omega), \quad \xi(t_0) = \tilde{x}_0(\omega)$$

with $x_0 \leq \tilde{x}_0$ respectively.

PROOF. According to the Lemma, we have

$$\Phi(w, t, \xi, w_0) \leq \tilde{\Phi}(w, t, \tilde{\xi}, w_0)$$

provided $\xi \leq \tilde{\xi}$. But by the same Lemma, we also have $\xi(t, \omega) \leq \tilde{\xi}(t, \omega)$ in the common interval where both sides are defined.

Combining these two inequalities and using Theorem 1 yields the desired result:

$$X(t) = \Phi(W(t), t, \xi(t), w_0) \leq \tilde{\Phi}(W(t), t, \tilde{\xi}(t), w_0).$$

Since the conditions imposed on σ are somewhat restrictive, we will weaken them by a method of smooth approximation.

For every $\varepsilon > 0$, let φ_ε be a modifier on $\mathbf{R}^3$, that is, $\varphi_\varepsilon \in C_0^\infty(\mathbf{R}^3)$, $\varphi_\varepsilon \geq 0$, $\int \varphi_\varepsilon = 1$ and the support of φ_ε is contained in the ball $B(0, \varepsilon)$. By extending its domain if necessary, we can assume that σ is defined and continuous in $\mathbf{R}^3$. Take $\sigma^\varepsilon = \sigma * \varphi_\varepsilon$ where $*$ denotes the convolution. Then, $\sigma^\varepsilon \in C^\infty(\mathbf{R}^3)$ and $\sigma^\varepsilon \rightarrow \sigma$ uniformly on every compact set in G as $\varepsilon \downarrow 0$.

Let Φ^ε be the unique solution to the problem

$$(5_\varepsilon) \quad \frac{d\Phi}{dw} = \sigma^\varepsilon(t, w, \Phi), \quad \Phi(w_0) = \xi$$

and g^ε be defined as in (6) except that Φ and σ are replaced by Φ^ε and σ^ε , respectively.

We are thus led to the following

THEOREM 3. *Assume that σ and b are continuous in G and locally Lipschitz continuous with respect to x . Let $X(t)$ and $\Phi(w, t, \xi, w_0)$ be the unique solutions to SDE (4) and problem (5), respectively. If there exist two functions $\tilde{\sigma}$ and $\tilde{g}$ satisfying the conditions stated in Theorem 2 except that inequality (10) is replaced by*

$$(10_\varepsilon) \quad g^\varepsilon(t, \xi, \omega) \leq \tilde{g}(t, \xi, \omega) \quad \text{in } D$$

for each sufficiently small positive number ε , then the conclusion of Theorem 2 holds.

PROOF. The existence and uniqueness of solution $X(t)$ to SDE (4) are obtained in [12]. Denote by $\varsigma(\omega)$ its “explosion time” and choose a sequence of compact sets $\{K_n\}$ such that $K_n \uparrow G$. If we define

$$\varsigma_n(\omega) = \inf\{t > t_0: (t, W(t, \omega), X(t, \omega)) \notin K_n\},$$

then each ς_n is a stopping time and $\varsigma_n \uparrow \varsigma$. Since b and σ^ε satisfy the conditions stated in Theorem 2, it follows that for $\varepsilon > 0$ sufficiently small, the solution $X^\varepsilon(t)$ to the SDE

$$(4_\varepsilon) \quad \begin{cases} dX(t) = b(t, W(t), X(t)) dt + \sigma^\varepsilon(t, W(t), X(t)) dW_t, \\ X(t_0) = x_0 \end{cases}$$

satisfies the inequality

$$(11_\varepsilon) \quad X^\varepsilon(t) \leq \tilde{\Phi}(W(t), t, \tilde{\xi}(t), w_0).$$

According to a theorem in [12], for every sequence $\{\varepsilon_m\}$ such that $\varepsilon_m \downarrow 0$ and every n , we have

$$\lim_{m \rightarrow \infty} E \left\{ \sup_{t \leq \varsigma_n} |X^{\varepsilon_m}(t) - X(t)|^2 \right\} = 0.$$

Hence, there exists a subsequence, again denoted by $\{\varepsilon_m\}$, such that

$$P \left\{ \lim_{m \rightarrow \infty} \sup_{t \leq \zeta_n} |X^{\varepsilon_m}(t) - X(t)| = 0 \right\} = 1.$$

Upon passage to limit in (11 _{ε}), we obtain the inequality (11) and the proof is complete.

Now we discuss some examples of its applications.

EXAMPLE 1. Consider two SDEs with the same diffusion coefficient σ :

$$\begin{cases} dX_i(t) = b_i(t, W(t), X_i(t)) dt + \sigma(t, W(t), X_i(t)) dW(t), \\ X_i(t_0) = x_0^i \quad (i = 1, 2) \end{cases}$$

where σ , b_1 and b_2 satisfy the conditions in Theorem 3 and $x_0^1 \leq x_0^2$, $b_1 \leq b_2$ in G .

By taking $\tilde{\sigma} = \sigma$ and

$$g_i^\varepsilon \equiv (\Phi_\xi^\varepsilon)^{-1} [b_i - \frac{1}{2}(\sigma_w^\varepsilon + \sigma_x^\varepsilon \sigma^\varepsilon) - \Phi_t^\varepsilon] \quad (i = 1, 2),$$

one can show without difficulty that

$$g_1^\varepsilon(t, \xi, \omega) \leq g_2^\varepsilon(t, \xi, \omega) \quad \forall \varepsilon > 0.$$

Hence, applying Theorems 1 and 2 to the equations

$$\begin{cases} dX_i^\varepsilon(t) = b_i(t, W(t), X_i^\varepsilon(t)) dt + \sigma^\varepsilon(t, W(t), X_i^\varepsilon(t)) dW(t), \\ X_i^\varepsilon(t_0) = x_0^i \quad (i = 1, 2), \end{cases}$$

we see that $X_1^\varepsilon(t) \leq X_2^\varepsilon(t)$ holds for $\forall \varepsilon > 0$ in the interval where they are both defined. Upon passage to limit as in the proof of Theorem 3, we obtain that $X_1(t) \leq X_2(t)$. This result is the most common version of comparison theorems considered by many authors (cf. [1, 3, 4, 7 and 10]).

EXAMPLE 2.¹ Consider two SDEs with different diffusion coefficients:

$$\begin{cases} dX_i(t) = \sigma_i(X_i(t)) dW(t) + \frac{1}{2}\sigma_i(X_i(t))\sigma_i'(X_i(t)) dt, \\ X_i(0) = x_i \quad (i = 1, 2) \end{cases}$$

where $\sigma_i > 0$ and $\sigma_i \in C^1(\mathbf{R})$ ($i = 1, 2$).

Let us first solve the following problems:

$$d\Phi_i/dw = \sigma_i(\Phi_i), \quad \Phi_i(0) = \xi_i \quad (i = 1, 2).$$

Clearly, the solutions $\Phi_i(w, \xi_i)$ will satisfy the equalities

$$\int_{\xi_1}^{\Phi_1(w, \xi_1)} \frac{dx}{\sigma_1(x)} = w = \int_{\xi_2}^{\Phi_2(w, \xi_2)} \frac{dx}{\sigma_2(x)}.$$

¹This example is adopted from [8].

In this case, the functions $g_i \equiv 0$, so the solutions to those problems corresponding to (7) are constants: $\xi_i \equiv x_i$. Hence, if for every $y \in \mathbf{R}$ the inequality $\int_{x_1}^y dx/\sigma_1(x) \geq \int_{x_2}^y dx/\sigma_2(x)$ holds, then $\Phi_1(w, x_1) \leq \Phi_2(w, x_2)$ and therefore

$$X(t) = \Phi(W(t), x) \leq \Phi(W(t), x) = X(t).$$

EXAMPLE 3. Consider the SDE

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))dW(t), \\ X(t_0) = x_0 \end{cases}$$

where $\sigma \in C^1$ with bounded derivative, $b \in C^0$ satisfying a local Lipschitz condition and a linearly bounded condition:

$$|b| \leq K_0(1 + x^2)^{1/2}.$$

Under these conditions, one can show that the functions σ and g defined in (6) are also linearly bounded, say,

$$|\sigma| \leq K_1(1 + x^2)^{1/2}, \quad |g| \leq K(1 + \xi^2)^{1/2}.$$

Taking $\tilde{\sigma} = \text{sgn}(w) \cdot K_1(1 + x^2)^{1/2}$ and $\tilde{g} = K(1 + \xi^2)^{1/2}$, we get $\tilde{\Phi} = \text{sh}(K_1|w| + c)$ where $c = \ln|\xi + \sqrt{1 + \xi^2}|$ and $\tilde{\xi} = \text{sh}(K(t - t_0) + c_1)$ where $c_1 = \ln|x_0 + \sqrt{1 + x_0^2}|$. Combining these two equalities and using Theorem 3, we obtain an asymptotic estimation for the paths of diffusion process $X(t)$,

$$X(t) \leq \text{sh}(K_1|W(t)| + K(t - t_0) + c_1).$$

A symmetric argument shows that

$$X(t) \geq \text{sh}(-K_1|W(t)| - K(t - t_0) + c_1).$$

REMARK. Theorems 1–3 can be extended to the case of continuous semimartingales without any difficulty. Moreover, if we drop the term $-\frac{1}{2}(\sigma_w + \sigma_x\sigma)$ in the definition of function g in (6), then we obtain solutions in the sense of Stratonovich. By using usual calculus instead of Itô's calculus, we can even assume that the driving process $W(t)$ is an arbitrary continuous process.

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