ON COHOMOLOGY AUTOMORPHISMS OF COMPLEX FLAG MANIFOLDS

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ABSTRACT. We conjecture a classification of the automorphisms of the rational cohomology ring of $U(n)/H$ for H a closed connected subgroup of maximal rank in $U(n)$, and prove a partial result.

1. Introduction. For any sequence $n_1, n_2, \dots, n_l$ of positive integers with sum n , let $F(n_1, \dots, n_l)$ denote the homogeneous space $U(n)/H$, where H is a subgroup of $U(n)$ conjugate to $U(n_1) \times U(n_2) \times \dots \times U(n_l)$. Then $F(n_1, \dots, n_l)$ can be given the structure of a complex manifold of complex dimension $\sum_{p < q} n_p n_q$. We can think of $F(n_1, \dots, n_l)$ as the space of orthogonal decompositions

$$\mathbb{C}^n = p_1 \oplus p_2 \oplus \dots \oplus p_l$$

with $\dim p_i = n_i$. Evidently there are canonical n_i -plane bundles over $F(n_1, \dots, n_l)$ for $1 \leq i \leq l$: we denote these bundles η_i and set

$$x_{i,h} = c_h(\eta_i) \in H^{2h}(F(n_1, \dots, n_l); \mathbb{Q}), \quad 1 \leq h \leq n_i.$$

The structure of $H^*(F(n_1, \dots, n_l); \mathbb{Q})$ is given by the following result [1].

THEOREM 1.1. *The cohomology ring $H^*(F(n_1, \dots, n_l); \mathbb{Q})$ is isomorphic to the quotient*

$$\mathbb{Q}[x_{i,h} | 1 \leq i \leq l, 1 \leq h \leq n_i] / (1 + x_{1,1} + \dots + x_{1,n_1}) \cdots (1 + x_{l,1} + \dots + x_{l,n_l}) = 1.$$

We make the following conjecture (cf. [4]).

CONJECTURE 1.2. *Any automorphism φ of $H^*(F(n_1, \dots, n_l); \mathbb{Q})$ has the form $\varphi(x_{i,h}) = m^h x_{\sigma(i),h}$ where $m \neq 0$ and σ is a permutation of $\{1, 2, \dots, l\}$ such that $n_p = n_q$ whenever $\sigma(p) = q$.*

REMARKS. 1. We can assume $n_1 \leq n_2 \leq \dots \leq n_l$.

2. Any $F(n_1, \dots, n_l)$ for which Conjecture 1.2 holds has trivial genus [5].

Conjecture 1.2 has been proved in the following special cases:

1. $l = 2$ [2, 6];
2. $l = 3$, $n_2 > n_1 = 1$ and $n_3 \geq 2n_2^2 - 1$ [4];
3. $n_1 = n_2 = \dots = n_{l-1} = 1$ [3].

In this paper we provide further evidence for Conjecture 1.2 by proving that any automorphism of $H^*(F(n_1, \dots, n_l); \mathbb{Q})$ acts on $H^2(F(n_1, \dots, n_l); \mathbb{Q})$ as expected (Theorem 3.2); this implies a partial result for all dimensions (Corollary 3.3). The proof is based on results of §2, where the height in $H^*(F(n_1, \dots, n_l); \mathbb{Q})$ of any two-dimensional cohomology class is determined.

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2. Heights of two-dimensional cohomology classes. Let 1^n denote 1 repeated n times. Then $H^*(F(1^n); \mathbf{Q})$ is generated by two-dimensional elements, which we denote by t_i (instead of $x_{i,1}$ as above). By Theorem 1.1, $H^*(F(1^n); \mathbf{Q})$ is the polynomial algebra $\mathbf{Q}[t_1, \dots, t_n]$ modulo the ideal generated by the elementary symmetric functions $e_i(t_1, \dots, t_n)$.

We denote the symmetric group on $\{1, 2, \dots, n\}$ by Σ_n . Let $P = \{P_1, P_2, \dots, P_l\}$ be a partition of $\{1, 2, \dots, n\}$, and put

$$\Sigma_P = \{\sigma \in \Sigma_n \mid \sigma(p) = q \text{ implies } p, q \in P_k, \text{ some } k\}.$$

If $\text{card } P_i = n_i$, there is a map $\pi_P: F(1^n) \rightarrow F(n_1, \dots, n_l)$ given by

$$\pi_P(l_1, l_2, \dots, l_n) = (p_1, p_2, \dots, p_l), \quad p_i = \bigoplus_{k \in P_i} l_k.$$

The symmetric group Σ_n acts on $H^*(F(1^n); \mathbf{Q})$ by permuting the t_i . For any subgroup G of Σ_n , let $H^*(F(1^n); \mathbf{Q})^G$ be the subring invariant under G . The next result shows that any cohomology ring $H^*(F(n_1, \dots, n_l); \mathbf{Q})$ with $n_1 + \dots + n_l = n$ can be regarded as an invariant subring of $H^*(F(1^n); \mathbf{Q})$.

PROPOSITION 2.1. *For any partition $P = \{P_1, \dots, P_l\}$ of $\{1, 2, \dots, n\}$ with $\text{card } P_i = n_i$, the map $\pi_P^*: H^*(F(n_1, \dots, n_l); \mathbf{Q}) \rightarrow H^*(F(1^n); \mathbf{Q})$ is an injection with image $H^*(F(1^n); \mathbf{Q})^{\Sigma_P}$.*

PROOF. There is a fibration

$$F(1^{n_1}) \times F(1^{n_2}) \times \dots \times F(1^{n_l}) \rightarrow F(1^n) \xrightarrow{\pi_P} F(n_1, \dots, n_l),$$

whose Serre spectral sequence collapses for degree reasons, so π_P^* injects. Now π_P^* maps $x_{i,h} \in H^*(F(n_1, \dots, n_l); \mathbf{Q})$ to $e_h(P_i)$, the h th elementary symmetric function in the variables $\{t_k \mid k \in P_i\}$. Since $\Sigma_P = \Sigma(P_1) \times \Sigma(P_2) \times \dots \times \Sigma(P_l)$, where $\Sigma(P_i)$ is the symmetric group on P_i , the conclusion follows.

For any homogeneous element u of $H^*(F(n_1, \dots, n_l); \mathbf{Q})$, we define the *height* of u to be the greatest power p so that $u^p \neq 0$. The next result gives an upper bound on height u for $u \in H^2(F(n_1, \dots, n_l); \mathbf{Q})$. (This upper bound will turn out to be exactly height u , as we show in Theorem 2.3.)

THEOREM 2.2. *Let*

$$u = a_1 x_{1,1} + a_2 x_{2,1} + \dots + a_l x_{l,1} \in H^2(F(n_1, \dots, n_l); \mathbf{Q}).$$

Define an equivalence relation E on $\{1, 2, \dots, l\}$ by $E = \{(p, q) \mid a_p = a_q\}$, and let $P(E) = \{S_1, S_2, \dots, S_r\}$ be the set of equivalence classes. Then

$$(1) \quad \text{height } u \leq \sum_{p < q} m_p m_q,$$

where

$$m_i = \sum_{k \in S_i} n_k, \quad 1 \leq i \leq r.$$

PROOF. Choose a partition $P = \{P_1, \dots, P_l\}$ of $\{1, 2, \dots, n\}$ with $\text{card } P_i = n_i$. By the preceding result, we can identify $H^*(F(n_1, \dots, n_l); \mathbf{Q})$ with $H^*(F(1^n); \mathbf{Q})^{\Sigma_P}$.

Then $u \in H^2(F(1^n); \mathbf{Q})$ has isotropy subgroup Σ_Q , where $Q = \{Q_1, \dots, Q_r\}$, $Q_i = \bigcup \{P_k \mid k \in S_i\}$. Since $\text{card } Q_i = m_i$, we have

$$u \in H^*(F(1^n); \mathbf{Q})^{\Sigma_Q} \cong H^*(F(m_1, \dots, m_r); \mathbf{Q})$$

and (1) follows since the right-hand side is the complex dimension of $F(m_1, \dots, m_r)$.

Now let u be as in the statement of the preceding result, and define a polynomial f in the a_i by

$$f(a_1, a_2, \dots, a_l) = \langle u^d, [F(n_1, \dots, n_l)] \rangle,$$

where $d = \sum_{p < q} n_p n_q$ and $\langle \cdot, \cdot \rangle$ denotes Kronecker product. We say u has *maximal height* in $H^*(F(n_1, \dots, n_l); \mathbf{Q})$ if $u^d \neq 0$.

THEOREM 2.3. *There is a constant $K \neq 0$ so that*

$$(2) \quad f(a_1, \dots, a_l) = K \prod_{p < q} (a_p - a_q)^{n_p n_q}.$$

In particular, u has maximal height in $H^(F(n_1, \dots, n_l); \mathbf{Q})$ if and only if all the a_i are distinct. It follows that (1) of Theorem 2.2 is an equality.*

PROOF. For $1 \leq p < q \leq l$, let

$$D_{pq} = \frac{\partial}{\partial(a_p - a_q)} = \frac{1}{2} \left(\frac{\partial}{\partial a_p} - \frac{\partial}{\partial a_q} \right).$$

Then $D_{pq}u = (x_{p,1} - x_{q,1})/2$. Apply D_{pq}^r to both sides of

$$f(a_1, \dots, a_l) = \langle u^d, [F(n_1, \dots, n_l)] \rangle$$

to get

$$D_{pq}^r f = \langle d(d-1) \cdots (d-r+1) u^{d-r} (D_{pq}u)^r, [F(n_1, \dots, n_l)] \rangle.$$

For $r < n_p n_q$ we have $u^{d-r} = 0$ when $a_p = a_q$, by 2.2. Thus $(a_p - a_q)^{n_p n_q}$ divides $f(a_1, \dots, a_l)$ for every $p < q$. Then either (2) holds or $f(a_1, \dots, a_l)$ is identically zero. But $F(n_1, \dots, n_l)$ can be given the structure of a Kähler manifold [3], so there must be *some* two-dimensional class of maximal height: thus, (2) holds. Then (1) must be an equality since the hypothesis of 2.2 implies that u has maximal height in $H^*(F(m_1, \dots, m_r); \mathbf{Q})$.

REMARK. A different proof of the formula for height u is given in [3].

3. Proof of the main theorem. To prove the main theorem we require a combinatorial lemma.

LEMMA 3.1. *Let $S \subset \{(p, q) \mid 1 \leq p < q \leq l\}$, $\bar{S}$ the equivalence relation generated by S , and $P(\bar{S})$ the partition of $\{1, 2, \dots, l\}$ into $\bar{S}$ -equivalence classes. If $\text{card } S = \binom{l-1}{2}$, then*

1. $P(\bar{S}) = \{\{1, 2, \dots, l\}\}$, or
2. $P(\bar{S}) = \{\{1, 2, \dots, k-1, k+1, \dots, l\}, \{k\}\}$ for some $1 \leq k \leq l$.

PROOF. Let $P(\bar{S}) = \{P_1, P_2, \dots, P_t\}$. Then

$$\text{card } S \leq \sum_{i=1}^t \binom{\text{card } P_i}{2} = \binom{l}{2} - \sum_{p < q} \text{card } P_p \text{ card } P_q.$$

By the hypothesis, this implies

$$\sum_{p < q} \text{card } P_p \text{ card } P_q \leq l - 1,$$

which can happen only if (1) or (2) holds.

Now we prove the main result.

THEOREM 3.2. *Let φ be an automorphism of $H^*(F(n_1, \dots, n_l); \mathbf{Q})$. Then*

$$\varphi(x_{i,1}) = mx_{\sigma(i),1}, \quad 1 \leq i \leq l,$$

where $m \neq 0$ and σ is a permutation of $\{1, 2, \dots, l\}$ such that $n_p = n_q$ whenever $\sigma(p) = q$.

PROOF. Let V denote the quotient of the free rational vector space on $x_{1,1}, x_{2,1}, \dots, x_{l,1}$ by $\mathbf{Q}(x_{1,1} + x_{2,1} + \dots + x_{l,1})$. We identify V with $H^2(F(n_1, \dots, n_l); \mathbf{Q})$, so φ is a linear automorphism of V which preserves heights. For $1 \leq p < q \leq l$, put

$$H_{pq} = \{a_1x_{1,1} + \dots + a_lx_{l,1} \in V \mid a_p = a_q\}.$$

Then each H_{pq} is a hyperplane in V , and

$$\bigcap_{p < q} (V - H_{pq}) = V - \bigcup_{p < q} H_{pq}$$

is precisely the set of vectors of maximal height. It follows that φ must permute the H_{pq} .

Let $S \subset \{(p, q) \mid 1 \leq p < q \leq l\}$. It is easy to see that

$$\bigcap_{(p,q) \in S} H_{pq} = \mathbf{Q}(x_{1,1} + \dots + x_{l,1}) = \{0\}$$

if and only if $P(\overline{S}) = \{\{1, 2, \dots, l\}\}$, in the notation of the lemma. Now if $1 \leq i \leq l$,

$$\mathbf{Q}x_{i,1} = \bigcap_{\substack{p < q \\ p, q \neq i}} H_{pq}.$$

Since φ permutes the H_{pq} , we have

$$\varphi(\mathbf{Q}x_{i,1}) = \bigcap_{(p,q) \in S} H_{pq}$$

for some S with $\text{card } S = \binom{l-1}{2}$. By the lemma, either

1. $P(\overline{S}) = \{\{1, 2, \dots, l\}\}$, or
2. $P(\overline{S}) = \{\{1, 2, \dots, k-1, k+1, \dots, l\}, \{k\}\}$ for some $1 \leq k \leq l$.

But (1) cannot hold, since φ cannot take a line to $\{0\}$. Thus (2) holds, and

$$\varphi(\mathbf{Q}x_{i,1}) = \bigcap_{\substack{p < q \\ p, q \neq k}} H_{pq} = \mathbf{Q}x_{k,1}.$$

It follows that

$$\varphi(x_{i,1}) = m_i x_{\sigma(i),1}, \quad 1 \leq i \leq l,$$

for some permutation σ of $\{1, 2, \dots, l\}$. Then

$$0 = \varphi(x_{1,1} + \dots + x_{l,1}) = m_1 x_{\sigma(1),1} + \dots + m_l x_{\sigma(l),1},$$

so $m_1 = m_2 = \cdots = m_l = m$ (clearly $m \neq 0$).

Finally, suppose $\sigma(p) = q$. Then

$$n_p(n - n_p) = \text{height } x_{p,1} = \text{height } x_{q,1} = n_q(n - n_q),$$

where we use the height formula. Since

$$n_p(n - n_p) - n_q(n - n_q) = \sum_{k \neq p, q} (n_p - n_q)n_k,$$

we have $n_p = n_q$ (unless $l = 2$, in which case the theorem is trivially true).

REMARK. In the preceding result it suffices to assume φ is an endomorphism of $H^*(F(n_1, \dots, n_l); \mathbf{Q})$ which is an automorphism on dimension 2. For if φ is an automorphism of $H^2(F(n_1, \dots, n_l); \mathbf{Q})$, it must take some class of maximal height to another class of maximal height (since the preimage of a class of maximal height has maximal height). Hence φ is nonzero on a generator of

$$H^{2d}(F(n_1, \dots, n_l); \mathbf{Q}), \quad d = \sum_{p < q} n_p n_q.$$

Then consideration of Poincaré duality shows that φ is an automorphism of $H^*(F(n_1, \dots, n_l); \mathbf{Q})$.

Theorem 3.2 yields the following result in higher dimensions.

COROLLARY 3.3. *Let D be the decomposables of $H^{2h}(F(n_1, \dots, n_l); \mathbf{Q})$, i.e. sums of products of positive-dimensional elements. Then for any automorphism φ of $H^*(F(n_1, \dots, n_l); \mathbf{Q})$ and i with $n_i \geq h$,*

$$\varphi(x_{i,h}) = m_h x_{\sigma(i),h} \pmod{D},$$

where $m_h \neq 0$ and σ is a permutation of $\{i \mid n_i \geq h\}$ such that $[n_p/h] = [n_q/h]$ whenever $\sigma(p) = q$.

PROOF. Let $\overline{D}$ be the ideal in $H^*(F(n_1, \dots, n_l); \mathbf{Q})$ generated by $\{x_{i,k} \mid 1 \leq i \leq l \text{ and } k \text{ does not divide } h\}$. Then $\varphi(\overline{D}) \subset \overline{D}$, so φ induces an endomorphism $\overline{\varphi}$ of $H^*(F(n_1, \dots, n_l); \mathbf{Q})/\overline{D}$. In fact $\overline{\varphi}$ is an automorphism, since it has inverse $\overline{\varphi}^{-1}$. Now there is an isomorphism

$$H^*(F(n_1, \dots, n_l); \mathbf{Q})/\overline{D} \xrightarrow{\sim} H^{*h}(F([n_1/h], \dots, [n_l/h]); \mathbf{Q}),$$

with the indicated shift in dimension. Apply 3.2 to $\overline{\varphi}$ to get the conclusion (note $H^{2h}(F(n_1, \dots, n_l); \mathbf{Q}) \cap \overline{D} = D$).

REMARK. If $n_p \geq 2n_{p-1}$ for each p , it follows that $\varphi(x_{i,h}) = m_h x_{i,h} \pmod{\text{decomposables}}$ for all i and h .

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