ON COLLECTIONWISE NORMALITY OF PRODUCT SPACES. II

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ABSTRACT. In this paper the following results will be obtained. (1) Let X be a closed image of a normal M -space and Y a paracompact first countable P -space. If $X \times Y$ is normal, then $X \times Y$ is collectionwise normal. (2) Let X be a collectionwise normal space and Y a σ -locally compact paracompact space. If $X \times Y$ is normal, then $X \times Y$ is collectionwise normal.

1. Introduction. Throughout this paper all spaces are Hausdorff spaces and all maps are continuous.

In general, when X and Y are collectionwise normal spaces, $X \times Y$ is not necessarily normal. Further, the following fact is known.

(I) (PRZYMUSIŃSKI [13]) ($MA + \neg CH$). There exists a paracompact first countable space X such that X^2 is normal but not collectionwise normal.

But, if we assume that $X \times Y$ is normal, then $X \times Y$ may be collectionwise normal. Concerning this, the following theorems are known.

(II) (RUDIN AND STARBIRD [15, 17]). *Let X be a collectionwise normal space and Y a paracompact M -space and $X \times Y$ normal. Then $X \times Y$ is collectionwise normal.*

(III) (NAGAMI [10]). *Let X be a collectionwise normal P -space and Y a paracompact σ -space and $X \times Y$ normal. Then $X \times Y$ is collectionwise normal.*

In this paper we shall prove the following results.

THEOREM 1. *Let X be a closed image of a normal M -space and Y a paracompact first countable P -space and $X \times Y$ normal. Then $X \times Y$ is collectionwise normal.*

THEOREM 2. *Let X be a collectionwise normal space and Y a σ -locally compact paracompact space and $X \times Y$ normal. Then $X \times Y$ is collectionwise normal.*

The definitions of M -spaces and P -spaces are due to Morita [7] and σ -spaces are due to Okuyama [12].

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2. Proof of Theorem 1. To prove Theorem 1, we use the following

(IV) [3]. Let X be a collectionwise normal Σ -space (in the sense of Nagami [11]) and Y a paracompact first countable P -space. Then $X \times Y$ is collectionwise normal.

We first prove the following lemmas. Lemma 1 is useful in proving collectionwise normality of a space X .

LEMMA 1. *Let $\{A_\lambda | \lambda \in \Lambda\}$ be a discrete family of closed subsets of a space X . If there exists a family $\{G_{\lambda,n} | \lambda \in \Lambda, n < \omega\}$ of open subsets of X such that*

(1) $\bigcup \{G_{\lambda,n} | n < \omega\} \supset A_\lambda$ for each $\lambda \in \Lambda$,

(2) $\text{Cl}(G_{\lambda,n}) \cap A_\mu = \emptyset$ if $\lambda \neq \mu$,

(3) $\{G_{\lambda,n} | \lambda \in \Lambda\}$ is discrete in X ,

then there exists a disjoint family $\{H_\lambda | \lambda \in \Lambda\}$ of open sets of X such that $H_\lambda \supset A_\lambda$ for each $\lambda \in \Lambda$. Here ω denotes the first infinite ordinal.

PROOF. Let us put $H_\lambda = \bigcup \{G_{\lambda,n} - \bigcup \{\text{Cl}(G_{\mu,i}) | i \leq n, \mu \in \Lambda, \mu \neq \lambda\} | n < \omega\}$. Then $\{H_\lambda | \lambda \in \Lambda\}$ satisfies the required conditions.

LEMMA 2. *Let X and Y be collectionwise normal spaces and C a closed discrete subspace of X and assume $X \times Y$ is normal. Let $\{A_\lambda | \lambda \in \Lambda\}$ be a discrete family of closed subsets of $X \times Y$. Then there exists a discrete family $\{G_\lambda | \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $G_\lambda \supset A_\lambda \cap (C \times Y)$ and $\text{Cl}(G_\lambda) \cap A_\mu = \emptyset$ if $\lambda \neq \mu$.*

PROOF.¹ Since $X \times Y$ is normal, there are open sets H_λ in $X \times Y$ such that $A_\lambda \subset H_\lambda$, $\text{Cl}(H_\lambda) \cap (\bigcup \{A_\mu | \mu \in \Lambda, \mu \neq \lambda\}) = \emptyset$. For each $x \in C$, there exists an open neighborhood $U(x)$ of x such that $\{U(x) | x \in C\}$ is discrete in X . Let us put $A(\lambda, x) = \{y \in Y | (x, y) \in A_\lambda\}$ for each $x \in C$. Then $\{A(\lambda, x) | \lambda \in \Lambda\}$ is a discrete family of closed subsets of Y . Since Y is collectionwise normal, there exist open sets $O(\lambda, x)$ in Y such that $O(\lambda, x) \supset A(\lambda, x)$ for each $\lambda \in \Lambda$ and $\{O(\lambda, x) | \lambda \in \Lambda\}$ is discrete in Y . Then $G_\lambda = H_\lambda \cap (\bigcup_{x \in C} U_x \times O(\lambda, x))$ has the desired properties.

THEOREM 3 (HOSHINA AND CHIBA).² *Let X, Y and Z be spaces such that $X \times Y$ is normal and $Y \times Z$ is collectionwise normal. Suppose Y is first countable and there is a closed onto map $f: Z \rightarrow X$ such that $X = \bigcup_{n < \omega} X_n$, X_n is closed discrete for $n \geq 1$ and $f^{-1}(x)$ is countably compact for each $x \in X_0$. Then $X \times Y$ is collectionwise normal.*

PROOF. It is easy to see that X and Y are collectionwise normal. To prove collectionwise normality of $X \times Y$, let $\{A_\lambda | \lambda \in \Lambda\}$ be a discrete family of closed subsets of $X \times Y$. Then, by Lemma 2, for each $n \geq 1$, there exists a discrete family $\{G_{\lambda,n} | \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $G_{\lambda,n} \supset A_\lambda \cap (X_n \times Y)$ and $\text{Cl}(G_{\lambda,n}) \cap A_\mu = \emptyset$ if $\lambda \neq \mu$.

Let us put $B_\lambda = A_\lambda - \bigcup_{n=1}^\omega G_{\lambda,n}$. Then B_λ is closed in $X \times Y$ and $B_\lambda \subset X_0 \times Y$. Let us put $\phi = f \times 1_Y: Z \times Y \rightarrow X \times Y$ and $F_\lambda = \phi^{-1}(A_\lambda)$ for each $\lambda \in \Lambda$. Then $\{F_\lambda | \lambda \in \Lambda\}$ is a discrete family of closed subsets of $Z \times Y$. Since $Z \times Y$ is

¹The referee has kindly pointed out this simple proof.

²T. Hoshina and the author proved this theorem independently.

collectionwise normal, there are open sets H_λ in $Z \times Y$ such that $F_\lambda \subset H_\lambda$ and $H_\lambda \cap H_\mu = \emptyset$ if $\lambda \neq \mu$.

Let us put $A(\lambda, x) = \{y \in Y | (x, y) \in A_\lambda\}$ for each $x \in X_0$. For each $y \in A(\lambda, x)$, there are an open set $W_y(x)$ in Z and an open neighborhood $V_x(y)$ of y in Y such that $f^{-1}(x) \subset W_y(x)$ and $W_y(x) \times V_x(y) \subset H_\lambda$ because $f^{-1}(x)$ is countably compact and Y is first countable. Since f is a closed map, there is an open neighborhood $U_y(x)$ of x in X such that $f^{-1}(U_y(x)) \subset W_y(x)$.

Now let us put

$$K_{\lambda,x} = \bigcup \{U_y(x) \times V_x(y) | y \in A(\lambda, x)\}$$

and

$$K_\lambda = \bigcup \{K_{\lambda,x} | x \in X_0\}.$$

Then we have:

- (1) K_λ is open in $X \times Y$.
- (2) $\phi^{-1}(K_\lambda) \subset H_\lambda$.
- (3) $A_\lambda \cap (X_0 \times Y) \subset K_\lambda$.

Moreover, it is easy to see that:

- (4) $K_\lambda \cap K_\mu = \emptyset$ if $\lambda \neq \mu$.
- (5) $K_\lambda \cap A_\mu = \emptyset$ if $\lambda \neq \mu$.

Since $B_\lambda \subset K_\lambda$ and $\{B_\lambda | \lambda \in \Lambda\}$ is a discrete family of closed subsets of $X \times Y$, there is a discrete family $\{G_{\lambda,0} | \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $B_\lambda \subset G_{\lambda,0}$, $\text{Cl}(G_{\lambda,0}) \subset K_\lambda$ because $X \times Y$ is normal. Then we have $\text{Cl}(G_{\lambda,0}) \cap A_\mu = \emptyset$ if $\lambda \neq \mu$ by (5). Let us consider $\{G_{\lambda,n} | 0 \leq n < \omega\}$. Then, by Lemma 1, there are open sets P_λ in $X \times Y$ such that $A_\lambda \subset P_\lambda$ for each $\lambda \in \Lambda$ and $P_\lambda \cap P_\mu = \emptyset$ if $\lambda \neq \mu$. Hence the proof of collectionwise normality of $X \times Y$ is complete.

REMARK 1. In Theorem 3, if we replace the condition “ $f^{-1}(x)$ is countably compact” by “ $f^{-1}(x)$ is compact”, then we can drop the first countability of Y .

PROOF OF THEOREM 1. Let Z be a normal M -space and $f: Z \rightarrow X$ a closed onto map and Y a paracompact first countable P -space such that $X \times Y$ is normal. Then $Z \times Y$ is collectionwise normal by (IV) because every normal M -space is a collectionwise normal Σ -space [11, and in 12, Theorem 3.3]. By Theorem 3.1 in [5], $X = \bigcup_{n < \omega} X_n$, X_n is closed discrete for $n \geq 1$ and $f^{-1}(x)$ is countably compact for each $x \in X_0$. Therefore, by Theorem 3, $X \times Y$ is collectionwise normal.

THEOREM 4. Let X be a collectionwise normal Σ -space and Y a closed image of a paracompact first countable M -space and $X \times Y$ is normal. Then $X \times Y$ is collectionwise normal.

PROOF. Let X be a collectionwise normal Σ -space and Z a paracompact first countable M -space and $f: Z \rightarrow Y$ a closed onto map such that $X \times Y$ is normal. Then $X \times Z$ is collectionwise normal by (IV) because every M -space is a P -space [7]. By [4], $Y = \bigcup_{n < \omega} Y_n$, Y_n is closed discrete for $n \geq 1$ and $f^{-1}(y)$ is compact for each $y \in Y_0$. Hence $X \times Y$ is collectionwise normal (see Remark 1).

We proved essentially the following in [2, Theorem 6].

(V). Let X be a locally countably compact space and Z a first countable space and $f: Z \rightarrow Y$ a closed map onto a space Y such that $Y = \bigcup_{n < \omega} Y_n$, Y_n is closed discrete for $n \geq 1$ and $f^{-1}(y)$ is compact for each $y \in Y_0$. If $X \times Z$ is normal, then $X \times Y$ is normal.

The following theorem can be proved in a way similar to the proof of Theorem 4 using (V).

THEOREM 5. *Let X be a locally countably compact collectionwise normal Σ -space and Y a closed image of a paracompact first countable M -space. Then $X \times Y$ is collectionwise normal.*

REMARK 2. Let X and Y be spaces in Theorem 1 or 4. It is not known whether $X \times Y$ is normal or not.

3. Proof of Theorem 2 and a related result. Even in case Y has only one nonisolated point, $X \times Y$ is not necessarily normal for a collectionwise normal space X . In fact, let X be the Dowker space constructed by Rudin [14] and $Y = [0, \omega]$ with the order topology. Then X is collectionwise normal and $X \times Y$ is not normal [14]. However, in case Y has only one nonisolated point, the following holds. “For any collectionwise normal space X , if $X \times Y$ is normal, then $X \times Y$ is collectionwise normal.”

We shall prove the following theorem which contains both in case (i) Y is σ -locally compact paracompact, and in case (ii) Y has only one nonisolated point.

THEOREM 6. *Let X be a collectionwise normal space and Y a paracompact space such that $Y = (\bigcup_{n < \omega} C_n) \cup D$ where C_n is a locally compact closed subspace of Y for each $n < \omega$ and for each $y \in D$ is an isolated point of Y . If $X \times Y$ is normal, then $X \times Y$ is collectionwise normal.*

There exist spaces expressed as the form of Y in Theorem 6 but neither (i) nor (ii). E. g., the Michael line [6] is so. Further, if Y is a paracompact subspace of F where F is the Bing's Example G or H [1], then Y satisfies the conditions in Theorem 6 but Y is necessarily neither (i) nor (ii). Therefore it seems good to consider the space Y satisfying the conditions in Theorem 6.

A subspace A of a space X is said to be P -embedded in X if every locally finite cozero-set cover of A has a refinement which can be extended to a locally finite cozero-set cover of X (Shapiro [16]).

LEMMA 3. *Let X and Y be collectionwise normal spaces and Y is expressed as $Y = C \cup D$ such that $C \cap D = \emptyset$ and each $y \in D$ is an isolated point of Y . Let $X \times Y$ be normal and $\{A_\lambda | \lambda \in \Lambda\}$ a discrete family of closed subsets of $X \times Y$. If there exists a disjoint family $\{G_\lambda | \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $G_\lambda \supset A_\lambda \cap (X \times C)$, then there exists a disjoint family $\{M_\lambda | \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $M_\lambda \supset A_\lambda$.*

PROOF. We can choose a discrete family $\{E_\lambda | \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $A_\lambda \cap (X \times C) \subset E_\lambda$, $\text{Cl}(E_\lambda) \subset G_\lambda - \bigcup_{\mu \neq \lambda} A_\mu$. Let us put $D_\lambda = A_\lambda - E_\lambda$. Then $\{D_\lambda | \lambda \in \Lambda\}$ is a discrete family of closed subsets of $X \times D$. Since $X \times D$ is collectionwise normal, there is a disjoint family $\{J_\lambda | \lambda \in \Lambda\}$ of open sets of $X \times D$

such that $D_\lambda \subset J_\lambda$. We remark that J_λ is open in $X \times Y$. Since D_λ and $\bigcup_{\mu \neq \lambda} \text{Cl}(E_\mu)$ are disjoint closed subsets of $X \times Y$ and $X \times Y$ is normal, there are open sets K_λ and L_λ in $X \times Y$ such that $D_\lambda \subset K_\lambda$, $\bigcup_{\mu \neq \lambda} \text{Cl}(E_\mu) \subset L_\lambda$ and $K_\lambda \cap L_\lambda = \emptyset$. Let us put $M_\lambda = E_\lambda \cup (J_\lambda \cap K_\lambda)$. Then M_λ is open in $X \times Y$. Moreover, it is easy to see that $M_\lambda \supset A_\lambda$ and $\{M_\lambda | \lambda \in \Lambda\}$ is disjoint.

PROOF OF THEOREM 6.³ Let X and Y be spaces in Theorem 6. Let $\{A_\lambda | \lambda \in \Lambda\}$ be a discrete family of closed subsets of $X \times Y$. For $n < \omega$, $X \times C_n$ is collectionwise normal by (II) in §1. Therefore, for each $n < \omega$, there is a discrete family $\{J_{\lambda,n} | \lambda \in \Lambda\}$ of open sets of $X \times C_n$ such that $A_\lambda \cap (X \times C_n) \subset J_{\lambda,n}$ for each $\lambda \in \Lambda$.

Also $X \times C_n$ is P -embedded in $X \times Y$ by Theorem 5.2 in [16] and Theorem 4 in [8]. Thus, by Theorem 2.4 in [9], there is a locally finite family $\{H'_{\lambda,n} | \lambda \in \Lambda\}$ of cozero-sets of $X \times Y$ such that

$$A_\lambda \cap (X \times C_n) \subset H'_{\lambda,n} \cap (X \times C_n) \subset J_{\lambda,n} \quad \text{for each } \lambda \in \Lambda.$$

Let $H_{\lambda,n}$ be open sets of $X \times Y$ such that $A_\lambda \cap (X \times C_n) \subset H_{\lambda,n}$, $\text{Cl}(H_{\lambda,n}) \subset H'_{\lambda,n}$. Let us put $L_{\lambda,n} = H_{\lambda,n} - \bigcup_{\mu \neq \lambda} \text{Cl}(H_{\mu,n})$. Then $L_{\lambda,n}$ is open in $X \times Y$, $A_\lambda \cap (X \times C_n) \subset L_{\lambda,n}$ for each $\lambda \in \Lambda$ and $\{L_{\lambda,n} | \lambda \in \Lambda\}$ is disjoint.

Since $X \times Y$ is normal, there is a discrete family $\{G_{\lambda,n} | \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $A_\lambda \cap (X \times C_n) \subset G_{\lambda,n}$, $\text{Cl}(G_{\lambda,n}) \subset L_{\lambda,n} - \bigcup_{\mu \neq \lambda} A_\mu$. If we consider $\{G_{\lambda,n} | \lambda \in \Lambda, n < \omega\}$, by Lemma 1, there is a disjoint family $\{G_\lambda | \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $G_\lambda \supset A_\lambda \cap [X \times (\bigcup_{n < \omega} C_n)]$. By Lemma 3, there exists a disjoint family $\{M_\lambda | \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $M_\lambda \supset A_\lambda$. Therefore $X \times Y$ is collectionwise normal. The proof of Theorem 6 is complete.

REFERENCES

1. R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. **3** (1951), 175–186.
2. K. Chiba, *On products of normal spaces*, Rep. Fac. Sci. Shizuoka Univ. **9** (1974), 1–11.
3. ———, *On collectionwise normality of product spaces*. I, Proc. Amer. Math. Soc. **91** (1984), 649–652.
4. V. V. Fillippov, *On feathered paracompacts*, Soviet Math. Dokl. **9** (1968), 161–164.
5. T. Ishii, *wM-spaces and closed maps*, Proc. Japan Acad. **46** (1970), 16–21.
6. E. Michael, *The product of a normal space and a metric space need not be normal*, Bull. Amer. Math. Soc. **69** (1963), 375–376.
7. K. Morita, *Products of normal spaces with metric spaces*, Math. Ann. **154** (1964), 365–382.
8. ———, *On the dimension of the product of topological spaces*, Tsukuba J. Math. **1** (1977), 1–6.
9. K. Morita and T. Hoshina, *P-embedding and product spaces*, Fund. Math. **93** (1976), 71–80.
10. K. Nagami, *σ -spaces and product spaces*, Math. Ann. **181** (1969), 109–118.
11. ———, *Σ -spaces*, Fund. Math. **65** (1969), 169–192.
12. A. Okuyama, *Some generalizations of metric spaces, their metrization theorems and product spaces*, Sci. Rep. Tokyo Kyoiku Daigaku **9** (1967), 236–254.
13. T. Przymusiński, *A Lindelöf space X such that X^2 is normal but not paracompact*, Fund. Math. **78** (1973), 291–296.
14. M. E. Rudin, *The normal space X for which $X \times I$ is not normal*, Fund. Math. **73** (1971), 179–186.
15. M. E. Rudin and M. Starbird, *Products with a metric factor*, Gen. Topology Appl. **5** (1975), 235–248.
16. H. L. Shapiro, *Extensions of pseudometrics*, Canad. J. Math. **18** (1966), 981–998.
17. M. Starbird, *The normality of products with a compact or a metric factor*, Ph. D. Thesis, University of Wisconsin, 1974.

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³The author first proved Theorem 6 by another method. T. Hoshina pointed out the proof can be made simpler by using the idea of P -embedding.