ESTIMATES FOR NONANALYTIC CUSP FORMS

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ABSTRACT. Estimates for the magnitude of nonanalytic cusp forms are obtained via the theory of the Selberg transformation and Fourier coefficient estimates.

1. Introduction. Let Γ be a finitely generated Fuchsian group of the first kind, acting on $\mathcal{H} = \{\tau/\tau = x + iy, y > 0\}$. Let $\mathcal{F}$ denote a fundamental domain of Γ with inequivalent parabolic cusps $s_1, \dots, s_\rho$. We consider the nonanalytic normalized eigenfunctions for Γ , solutions of the eigenvalue problem

$$(1.1) \quad y^2(u_{xx} + u_{yy}) + \lambda u = 0$$

with $u \in L_2(\Gamma|\mathcal{H})$, the Γ -invariant functions on $\mathcal{H}$ with finite norm given by the scalar product

$$\langle f, g \rangle \doteq \int_{\mathcal{F}} f(\tau) \overline{g(\tau)} (d\tau), \quad (d\tau) = y^{-2} dx dy.$$

We assume further that our eigenfunctions are cusp forms. This requires that $\lambda = \lambda_j = t_j(1 - t_j)$, $t_j = 1/2 + ir_j$, $j = 1, 2, 3, \dots$, belong to the discrete spectrum of the non-Euclidean Laplace operator in (1.1). These eigenfunctions, $e_j(\tau)$, may be characterized as follows. Let A_i , $i = 1, \dots, \rho$, be chosen in $SL_2(\mathbf{R})$ so that $A_i s_i = \infty$, $A_i \Gamma_i A_i^{-1} = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbf{Z} \right\}$, where $\Gamma_i \subset \Gamma$ is the stabilizer group of s_i . If $\tau_k = x_k + iy_k = A_k \tau$ for $k = 1, \dots, \rho$, then each cusp form has expansions of the form

$$(1.2) \quad e_j(\tau) = \sum_{n \neq 0} \beta_{n,j}^k y_k^{1/2} K_{ir_j}(2\pi|n|y_k) e^{2\pi i n x_k},$$

where K_{ir} is the Bessel function. The cusp forms are characterized among the $L_2(\Gamma|\mathcal{H})$ eigenfunctions by the vanishing of the constant (in x_k) term in the Fourier expansion at all cusps. Aside from some special examples for congruence subgroups of the modular group [3, p. 146], no cusp forms are explicitly known.

In what follows, $A \ll B$, $A = O(B)$ will mean $|A| \leq \text{const}|B|$ and $C(\dots)$ will indicate various constants depending on the indicated parameters. It is clear from the expansions (1.2) that $e_j(\tau) \rightarrow 0$ as $\tau \rightarrow s_i$ in F . It follows that $e_j(\tau)$ is bounded on $\mathcal{F}$. A bound of the form

$$|e_j(\tau)| \ll (1 + r_j^2),$$

which is uniform in τ , follows from the theory of the resolvent kernel for the Laplace operator (see [5, p. 274]). This paper contains improvements on the results given above.

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Select $C > 0$ large enough so that for any τ in $\mathcal{F}$ there is at most one i so that $y_i \geq C$. Define $\kappa = \kappa(\tau)$ as follows:

$$(1.3) \quad \kappa = \begin{cases} i & \text{if } y_i \geq C, \\ 1 & \text{if } y_i < C \text{ for } 1 \leq i \leq \rho. \end{cases}$$

It is clear that there is $C_0 > 0$ so that $y_\kappa \geq C_0$ for any τ in $\mathcal{F}$.

THEOREM 1. *If $1/2 < \alpha \leq 3/4$ and $\kappa = \kappa(\tau)$, then*

$$|e_j(\tau)| < C(\alpha)(1 + r_j^{2\alpha})y_\kappa^{3/2-2\alpha}.$$

In particular, the bound $e_j(\tau) \ll (1 + r_j^{3/2})$ holds uniformly in τ , while for τ fixed

$$|e_j(\tau)| < C(\tau, \varepsilon)(1 + r_j^{1+\varepsilon})$$

holds for any given $\varepsilon > 0$.

The following inequality is better if τ approaches a cusp.

THEOREM 2. *If $y_\kappa > r_j/8$ and $\kappa = \kappa(\tau)$, then*

$$e_j(\tau) \ll e^{\pi r_j/2} r_j^{1/2} e^{-2\pi y_\kappa} y_\kappa^{-1/2}.$$

Theorem 2 is the result of a simple estimate for the coefficients in the expansions (1.2). The estimate, which may be of some independent interest, is given as Theorem 3 in the next section. The first theorem results from using Selberg's theory of point pair invariants to construct a kernel with a stronger singularity and large eigenvalues than the resolvent kernel.

2. Poincaré series. In this section we state results about Poincaré series and Eisenstein series that will be needed later.

For $M \in \Gamma$ define

$$\eta_{l,M} = x_{l,M} + iy_{l,M} \doteq A_l M A_l^{-1} \eta_l = A_l M \tau$$

for $l = 1, \dots, \rho$. The Poincaré series for the l th cusp are defined by

$$(2.1) \quad U_n^l(\tau; s) \doteq \sum_{\Gamma_l \backslash \Gamma} e^{-2\pi |n|y_{l,M} + 2\pi i n x_{l,M}} y_{l,M}^s$$

for $\sigma > 1$, $s = \sigma + it$. The sum is over representatives of left Γ_l cosets of Γ . The Eisenstein series for the l th cusp is defined to be $U_0^l(\tau; s)$. The Eisenstein series has an expansion at the k th cusp of the form (see [4, p. 24])

$$(2.2) \quad \begin{aligned} U_0^l(\tau; s) &= \delta_{k,l} y_k^s + \phi_k^l(s) y_k^{1-s} \\ &+ 2\pi^\Gamma \Gamma(s)^{-1} \sum_{m \neq 0} |m|^{s-1/2} L_{k,m}^l(s) K_{s-1/2}(2\pi |m| y_k) e^{2\pi i m x_k}, \end{aligned}$$

where

$$\delta_{k,l} = \begin{cases} 1, & k = l, \\ 0, & k \neq l, \end{cases}$$

$K_{s-1/2}$ is the Bessel function and $\Gamma(s)$ is the gamma function. The coefficients are given by

$$L_{k,m}^l(s) = \sum_{(c,d)} e^{2\pi i m d/c} c^{-2s}, \quad \sigma > 1,$$

$$\phi_k^l(s) = \sqrt{\pi} \Gamma(s - \frac{1}{2}) \Gamma(s)^{-1} L_{k,0}^l(s),$$

where $\begin{pmatrix} * & * \\ c & d \end{pmatrix} = A_l^{-1} M A_k^{-1}$, with M from $\Gamma_l \setminus \Gamma$. The Bessel function satisfies

$$(2.3) \quad K_{s-1/2}(x) \ll x^{-1/2} e^{-x}$$

uniformly for $1 \leq s \leq 2$ or for $x \gg t$, where $s = 1/2 + it$. It is known from the theory of Eisenstein series that $L_{k,0}^l(s)$ has at worst a simple pole at $s = 1$. Thus for $1 < s \leq 2$

$$|L_{k,m}^l(s)| \leq L_{k,0}^l(s) \ll (s-1)^{-1}.$$

It follows from this, (2.2) and (2.3) that

$$(2.4) \quad \begin{aligned} U_0^l(\tau; s) - \delta_{k,l} y_k^s &\ll \delta_{k,l} y_k^s - \delta_{k,l} y_k^s + (s-1)^{-1} y_k^{1-s} \\ &+ (s-1)^{-1} \sum_{m \neq 0} |m|^{s-1/2} |K_{s-1/2}(2\pi|m|y_k)| \\ &\ll (s-1)^{-1} \quad (\kappa = \kappa(\tau)) \end{aligned}$$

uniformly for $1 < s \leq 2$, and $\tau \in \mathcal{F}$.

We can now establish the estimate for the Fourier coefficients of the cusp forms.

THEOREM 3. $\beta_{m,j}^l \ll e^{\pi r_j/2} (r_j + 1)^{1/2} |m|^{1/2} \log(|m| + 5)$.

PROOF. The Petersson scalar product formula gives

$$\begin{aligned} \langle U_m^l(\tau; s), e_j(\tau) \rangle &= \bar{\beta}_{m,j}^l \pi^{1/2} (\pi|m|)^{1/2-s} 2^{1-2s} \\ &\quad \times \Gamma(s - ir_j - 1/2) \Gamma(s + ir_j - 1/2) \Gamma(s)^{-1} \end{aligned}$$

(see [4, p. 57]); note that in [4] the Poincaré series differ from the U_n^l by a factor depending on s). Pick $s = 1 + 1/\log(|m| + 5) + ir_j$, apply Stirling's estimate and the Schwarz inequality. It follows that

$$\beta_{m,j}^l \ll |m|^{1/2} e^{\pi r_j/2} (r_j + 1)^{1/2} \langle U_m^l, U_m^l \rangle^{1/2}.$$

Since by (2.4)

$$\begin{aligned} |U_m^l(\tau, s)| &\leq e^{-2\pi y_l} y_l^\sigma + \sum_{M \neq \text{id}} y_{l,M}^\sigma \\ &= e^{-2\pi y_l} y_l^\sigma + (U_0^l(\tau'; \sigma) - y_l^\sigma) \ll (\sigma - 1)^{-1}, \end{aligned}$$

we have $\langle U_m^l, U_m^l \rangle \ll \log^2(|m| + 5)$ with the choice of s given above.

Theorem 2 is an immediate consequence of Theorem 3, (2.3) and (1.2).

3. Estimation of the point pair invariant. Selberg's theory shows us that the e_j satisfy integral equations of the form

$$h(r_j) e_j(\tau) = \int_{\mathcal{H}} k(\tau, \tau') e_j(\tau') (d\tau')$$

provided that k is a point pair invariant. This means that k is a function of the hyperbolic distance $d(\tau, \tau')$, satisfying suitable conditions of growth and regularity. If (see [6, p. 72])

$$t(\tau, \tau') \doteq |\tau - \tau'|^2 / yy' = 2 \cosh d(\tau, \tau') - 2,$$

then h and k are related by

$$\begin{aligned}
 (3.1) \quad g(u) &\doteq \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iru} h(r) dr, \\
 g(u) &\doteq Q(w), \quad w = 2 \cosh u - 2, \\
 k(t) &\doteq -\frac{1}{\pi} \int_t^\infty (w-t)^{-1/2} dQ(w), \\
 k(\tau, \tau') &= k(t(\tau, \tau')).
 \end{aligned}$$

We will construct k by choosing $h(r) = (4 + r^2)^{-p-1/2}$ with $0 < p < 1/2$. This gives, by (3.1),

$$(3.2) \quad g(u) = 4^{-p} \pi^{-1/2} \Gamma(p + 1/2)^{-1} u^p K_p(2u),$$

where Γ is the gamma function and K_p is the Bessel function. The series expansion K_p in powers of u gives

$$u^p K_p(2u) = \pi \csc \pi p \{ \Gamma(1-p)^{-1} - \Gamma(p+1) u^{2p} + \dots \}.$$

Termwise differentiation shows

$$(3.3) \quad g'(u) \ll u^{2p-1}, \quad g''(u) \ll u^{2p-2}$$

for $u \rightarrow 0$. The asymptotic estimate for K_p implies

$$(3.4) \quad g'(u) \ll u^{p-1/2} e^{-2u}, \quad g''(u) \ll u^{p-1/2} e^{-2u}$$

for $u \rightarrow \infty$.

The relation between u and w in (3.1) gives

$$(3.5) \quad u = \begin{cases} w^{1/2} + O(w), & w \rightarrow 0, \\ \log(1+w + O(w^{-1})), & w \rightarrow \infty. \end{cases}$$

It follows that

$$\begin{aligned}
 (3.6) \quad Q'(w) &\ll \begin{cases} w^{p-1}, & 0 < w < 1, \\ (1+w)^{-11/4}, & w \geq 1, \end{cases} \\
 Q''(w) &\ll \begin{cases} w^{p-2}, & 0 < w < 1, \\ (1+w)^{-15/4}, & w \geq 1. \end{cases}
 \end{aligned}$$

It is now possible to estimate the function k , where by (3.1)

$$k(t) = -\frac{1}{\pi} \int_0^\infty Q'(w+t) w^{-1/2} dw.$$

If $0 < t < 1$, then by (3.6)

$$\begin{aligned}
 k(t) &\ll t^{p-1/2} \int_0^{t^{-1}} (w+1)^{p-1} w^{-1/2} dw + \int_1^\infty (1+w+t)^{-2} w^{-1/2} dw \\
 &\ll t^{p-1/2} + t^{p-1/2} \int_1^{t^{-1}} w^{p-3/2} dw \ll t^{p-1/2}.
 \end{aligned}$$

If $t \geq 1$ then

$$k(t) \ll \int_0^\infty (1+w+t)^{-11/4} w^{-1/2} dw \ll t^{-2}.$$

A similar estimate for k' can be obtained in the same way from (3.6). In summary, if

$$k_1(t) \doteq \begin{cases} t^{p-1/2}, & 0 < t < 1, \\ t^{-2}, & t \geq 1, \end{cases}$$

$$k'_1(t) \doteq \begin{cases} t^{p-3/2}, & 0 < t < 1, \\ t^{-3}, & t \geq 1, \end{cases}$$

then the following lemma holds.

LEMMA 1. $k(t) \ll k_1(t)$, $k'(t) \ll k'_1(t)$.

4. Lemmas for Theorem 1. Define

$$(4.1) \quad K(\tau, \tau') \doteq \sum_{\gamma \in \Gamma} k(\tau, \gamma\tau') = \sum_{\Gamma_i \setminus \Gamma} \sum_{n=-\infty}^{+\infty} k(\tau_i, U^n \tau_{i,M}),$$

$$k_0(\tau, \tau') \doteq \int_{-\infty}^{+\infty} k(\tau, U^x \tau') dx,$$

$$K_i(\tau, \tau') \doteq \sum_{\Gamma_i \setminus \Gamma} k_0(\tau_i, \tau_{i,M}), \quad \tau = x + iy, \tau' = x' + iy',$$

and $i = 1, \dots, \rho$. The sums indicated by $\Gamma_i \setminus \Gamma$ are over all cosets, $\Gamma_i M$, and $U^x \doteq \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. We seek to estimate the kernel K , with the contribution from K_κ removed. The following lemmas furnish preparation for this estimate.

LEMMA 2. $k_0(\tau, \tau') \ll y^{-1/8} y'^{9/8}$.

PROOF. If $7y/8 \leq y' \leq 9y/8$, then Lemma 1 implies

$$k_0(\tau, \tau') \ll (yy')^{1/2} \int_{-\infty}^{+\infty} k_1(t^2) dt \ll (yy')^{1/2} \ll y^{-1/8} y'^{9/8}.$$

Now assume $y' < 7y/8$ or $9y/8 < y'$. It follows that $(y - y')^2 / yy' > 1/72$, and now by Lemma 1,

$$k(\tau, \tau') \ll (|\tau - \tau'| / yy')^{-2}.$$

This gives

$$k_0(\tau, \tau') \ll (yy')^2 \int_{-\infty}^{+\infty} \{(x - x' - t)^2 + (y - y')^2\}^{-2} dt \ll (yy')^2 |y - y'|^{-3}.$$

If $y' > 9y/8$ then

$$(y'y)^2 |y - y'|^{-3} = y^2 y'^{-1} |1 - y/y'|^{-3} \ll y^2 y'^{-1} \ll y^{-1/8} y'^{9/8}.$$

If $y' < 7y/8$ then similarly

$$y^{-1} y'^2 |1 - y'/y|^{-3} \ll y^{-1} y'^2 \ll y^{-1/8} y'^{9/8}.$$

This completes the proof of Lemma 2.

LEMMA 3. *If $7y/8 \leq y' \leq 9y/8$ and $y \neq y'$ then*

$$\sum_{n=-\infty}^{+\infty} k(\tau, U^n \tau) \ll y^{-1/8} y'^{9/8} + (yy')^{1/2-p} \sum_{|n| \leq 3} \{(x - x' - n)^2 + (y - y')^2\}^{p-1/2}.$$

If $y' < 7y/8$ or $y' > 9y/8$ then

$$\sum_{n=-\infty}^{+\infty} k(\tau, U^n \tau') \ll y^{-1/8} y'^{9/8}.$$

PROOF. Let $7y/8 \leq y' \leq 9y/8$ and $y \neq y'$. Lemma 1 implies

$$\begin{aligned} \sum_n k(\tau, U^n \tau') &\ll (yy')^{1/2-p} \sum_{|m| \leq 3} \{(x - x' - m)^2 + (y - y')^2\}^{p-1/2} \\ &\quad + (yy')^{1/2-p} \sum_{1 \leq |x-x'-m| \leq (yy')^{1/2}} |m|^{2p-1} \\ &\quad + (yy')^2 \sum_{|x-x'-m| > (yy')^{1/2}} |m|^{-4}. \end{aligned}$$

The second and third terms on the right are each

$$\leq (yy')^{1/2} \ll y^{-1/8} y'^{9/8}.$$

If $y' < 7y/8$ or $y' > 9y/8$, Lemma 1 implies that

$$\begin{aligned} \sum_n k(\tau, U^n \tau') &\ll (yy')^2 \int_{-\infty}^{+\infty} \{(x - x' - t)^2 + (y - y')^2\}^{-2} dt \\ &\ll (yy')^2 |y - y'|^{-3}. \end{aligned}$$

As in the proof of Lemma 2, this term satisfies the stated inequality.

Finally we consider the difference $D(\tau, \tau')$ defined by

$$\begin{aligned} (4.2) \quad D(\tau, \tau') &\doteq \sum_{n=-\infty}^{+\infty} k(\tau, U^n \tau') - k_0(\tau, \tau') \\ &= \sum_n \int_0^1 \{k(\tau, U^n \tau') - k(\tau, U^{n+t} \tau')\} dt. \end{aligned}$$

The following lemma contains the estimate for D that we require.

LEMMA 4. *If $7y/8 \leq y' \leq 9y/8$ and $y \neq y'$ then*

$$\begin{aligned} D(\tau, \tau') &\ll (yy')^{1/2-p} + (yy')^{1/2-p} \sum_{|n| \leq 3} \{(y - y')^2 + (x - x' - n)^2\}^{p-1/2} \\ &\quad + (yy')^{1/2-p} \sum_{|n| \leq 3} \int_0^1 \{(y - y')^2 + (x - x' - n - t)^2\}^{p-1/2} dt. \end{aligned}$$

If $y' < 7y/8$ or $y' > 9y/8$ then $D(\tau, \tau') \ll 1$.

PROOF. Assume $7y/8 \leq y \leq 9y/8$ and $y \neq y'$. If $|n| \geq 3$ then

$$|k(\tau, U^n \tau') - k(\tau, U^{n+t} \tau')| \ll k'(\xi)(yy')^{-1}|x - x' - n|,$$

where $\xi = (yy')^{-1}\{(y - y')^2 + (x - x' - n - \eta)^2\}$ with $0 \leq \eta \leq 1$. By Lemma 1 the term on the right is

$$\ll k'_1(\xi)(yy')^{-1}|x - x' - n|.$$

If $3 \leq |n| \leq (yy')^{1/2}$ then this bound is $\ll (yy')^{1/2-p}|n|^{2p-2}$. If $|n| > (yy')^{1/2}$, then it is $\ll (yy')|n|^{-5}$. The terms with $|n| \leq 3$ are estimated directly with Lemma 1. Since $p < 1/2$, summing over n gives the stated inequality.

If $y' < 7y/8$ or $y' > 9y/8$ then Lemma 1 implies that

$$k'(\xi)(yy')^{-1}|x - x' - n| \ll |n|\{(y - y')^2 + n^2\}^{-3}(yy')^2$$

for $|n| \geq 3$. The estimate for D in this case is

$$\begin{aligned} D &\ll (yy')^2(y - y')^{-4} + (yy')^2 \sum_{|n| \geq 3} |n|\{(y - y')^2 + n^2\}^{-3} \\ &\ll (yy')^2(y - y')^{-4} \ll 1. \end{aligned}$$

This completes the proofs of the required lemmas for Theorem 1.

5. Estimation of cusp forms. Define the kernel

$$(5.1) \quad K^*(\tau, \tau') = K(\tau, \tau') - K_\kappa(\tau, \tau'),$$

where K and K_i are defined in (4.1) and $\kappa = \kappa(\tau)$ as given in (1.4). We have

$$K_\kappa(\tau, \tau') = \sum_{\Gamma_\kappa \backslash \Gamma} k_0(\tau_\kappa, \tau'_{\kappa, M}),$$

where by Lemma 2 each term has the majorization

$$(5.2) \quad k_0(\tau_\kappa, \tau'_{\kappa, M}) \ll y_{\kappa, M}^{9/8}.$$

This estimate allows

$$\begin{aligned} \int_{\mathcal{F}} K_\kappa(\tau, \tau') e_j(\tau') (d\tau') &= \sum_{\Gamma_i \backslash \Gamma} \int_{\mathcal{F}} k_0(\tau_\kappa, \tau'_{\kappa, M}) e_j(\tau') (d\tau') \\ &= \int_0^\infty y'_\kappa dy'_\kappa \int_0^1 k_0(\tau_\kappa, \tau'_\kappa) e_j(\tau') dx'_\kappa. \end{aligned}$$

Since $k_0(\tau_\kappa, \tau'_\kappa)$ does not depend on x'_κ and the constant term of the expansion of the e_j at the κ th cusp is zero, it follows that the integral above is zero. Therefore

$$h(r_j) e_j(\tau) = \int_{\mathcal{F}} K^*(\tau, \tau') e_j(\tau') (d\tau'),$$

hence

$$|e_j(\tau)|^2 \leq |h(r_j)|^2 \int_{\mathcal{F}} |K^*(\tau, \tau')|^2 (d\tau').$$

We will estimate the integral above. For τ given, and $\kappa = \kappa(\tau)$, set $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, with

$$\mathcal{F}_1 \doteq \{\tau' \in \mathcal{F} | y'_\kappa \geq C_0/2\}, \quad \mathcal{F}_2 \doteq \{\tau' \in \mathcal{F} | y'_\kappa < C_0/2\}.$$

Also let

$$S_\tau = \{\tau' \in \mathcal{F} | 7y_\kappa/8 \leq y'_\kappa \leq 9y_\kappa/8\}.$$

If $\tau' \in F_2$, then from Lemma 3

$$K(\tau, \tau') = \sum_{\substack{M \\ \Gamma_\kappa \setminus \Gamma}} \left(\sum_n k(\tau_\kappa, U^n \tau'_{\kappa, M}) \right) \ll \sum_{\Gamma_\kappa \setminus \Gamma} y'_{\kappa, M}{}^{9/8} \ll 1.$$

The same estimate for K_κ follows from (5.2), thus

$$(5.3) \quad K^*(\tau, \tau') \ll 1 \quad \text{for } \tau' \in \mathcal{F}_2.$$

If $\tau' \in \mathcal{F}_1$, write

$$(5.4) \quad \begin{aligned} |K^*(\tau, \tau')|^2 &\ll |D(\tau_\kappa, \tau'_\kappa)|^2 + \left| \sum_{M \neq \text{id}} \sum_n k(\tau_\kappa, U^n \tau'_{\kappa, M}) \right|^2 + \left| \sum_{M \neq \text{id}} k_\kappa(\tau, \tau') \right|^2 \\ &\doteq |D|^2 + \left| \sum_1 \right|^2 + \left| \sum_2 \right|^2. \end{aligned}$$

Lemma 2 and (2.4) show

$$(5.5) \quad \sum_2 \ll \sum_{M \neq \text{id}} y'_{\kappa, M}{}^{9/8} \ll 1$$

for $\tau' \in \mathcal{F}$. Lemma 3 and (2.4) give

$$(5.6) \quad \sum_1 \ll 1 + y_\kappa^{1-2p} \sum_{\substack{7y_\kappa/8 \leq y_{\kappa, M} \leq 9y_\kappa/8 \\ M \neq \text{id}}} \sum_{|n| \leq 3} \{(x_\kappa - x'_{\kappa, M} - n)^2 + (y_\kappa - y'_{\kappa, M})^2\}^{p-1/2}.$$

Note that the first sum on the right of (5.6) can contain at most finitely many terms since $y_\kappa \geq C_0 > 0$. The number of terms is bounded above by a constant that is independent of y'_κ . If $y_\kappa \geq C_1$, for some sufficiently large $C_1 > 1$ then the sum contains no terms. It follows that

$$(5.7) \quad \int_{\mathcal{F}} \left| \sum_1 \right|^2 (d\tau') \ll 1 + C_1^2 \sum \sum \int_{A_\kappa M \mathcal{F}_1} \{(x_\kappa - x'_\kappa - n)^2 + (y_\kappa - y'_\kappa)^2\}^{2p-1} (d\tau'_\kappa).$$

The integrals on the right are finite if $p > 0$. It follows that

$$(5.8) \quad \int_{\mathcal{F}_1} |K^*|^2 (d\tau') \ll 1 + \int_{\mathcal{F}_1} |D|^2 (d\tau').$$

Lemma 4 shows

$$(5.9) \quad \int_{\mathcal{F}_1 \setminus S_\tau} |D|^2 (d\tau') \ll 1$$

and

(5.10)

$$\begin{aligned}
& \int_{S_\tau} |D(\tau, \tau')|^2 (d\tau') \ll y_\kappa^{-4p} \int_0^1 dx'_\kappa \int_{7y_\kappa/8}^{9y_\kappa/8} dy'_\kappa \\
& + y_\kappa^{-4p} \sum_{|n| \leq 3} \int_0^1 dx'_\kappa \int_{7y_\kappa/8}^{9y_\kappa/8} \{(y_\kappa - y'_\kappa)^2 + (x_\kappa - x'_\kappa - n)^2\}^{2p-1} dy'_\kappa \\
& + y_\kappa^{-4p} \sum_{|n| \leq 3} \int_0^1 dt \int_0^1 dx'_\kappa \int_{7y_\kappa/8}^{9y_\kappa/8} \{(y_\kappa - y'_\kappa)^2 + (x_\kappa - x'_\kappa - n - t)^2\}^{2p-1} dy'_\kappa \\
& < C(p) y_\kappa^{1-4p}.
\end{aligned}$$

It follows from (5.2), (5.4), (5.5) and (5.7)–(5.9) that

$$|e_j(\tau)|^2 < C(p)(1 + r_j^2)^{2p+1}(1 + y_\kappa^{1-4p}).$$

Setting $\alpha = p + 1/2$ and again noting that $y_\kappa \geq C_0$ gives Theorem 1 as stated.

6. Remarks. It might be possible to improve the estimate of Theorem 1 by choosing $h(\tau)$ to be large locally near the desired value of r_j . I have not been able to do as well as Theorem 1 with this approach.

It is known, via the methods in [1], that it is not possible to replace the factor $e^{\pi r_j/2}$ in Theorem 3 by e^{ar_j} with $a < \pi/2$.

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