A WEIGHTED POLYNOMIAL INEQUALITY

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ABSTRACT. In the theory of orthogonal polynomials for weights with noncompact support, much use is made of inequalities relating weighted integrals of polynomials over infinite and finite ranges. Using a short new method of proof, we show such inequalities hold for very general weights in L_p and certain Orlicz spaces.

Introduction. Let $\mathbf{P}_n$ denote the class of real polynomials of degree at most n , and let $C_1, C_2, \dots$ denote positive constants independent of n and $P \in \mathbf{P}_n$. Freud made extensive use of the following inequality [3, Theorem 2.6]:

$$\|PW\|_{L_2(-\infty, \infty)} \leq (1 + C_1 e^{-C_2 n}) \|PW\|_{L_2(-C_3 q_{2n}, C_3 q_{2n})},$$

$P \in \mathbf{P}_n$. Here $W = \exp(-Q)$ where Q and Q' satisfy certain restrictions and q_n is the positive root of $q_n Q'(q_n) = n$ and of (4) below. The corresponding L_∞ inequality for the Hermite weight $W = \exp(-x^2/2)$ appears in [2]. Similar inequalities for special weights have frequently been used by Freud and Nevai.

For the Hermite weight, $1 \leq p \leq \infty$ and $0 < r < \infty$, Zalik [6, Theorems 1 and 4] proved

$$(1) \quad \|x^r P(x) W(x)\|_{L_p(-\infty, \infty)} \leq C_4 n^{r/2} \|P(x) W(x)\|_{L_p(-\infty, \infty)},$$

$$(2) \quad \|PW\|_{L_p(-\infty, \infty)} \leq C_5 \|PW\|_{L_p(-4\sqrt{n}, 4\sqrt{n})},$$

$P \in \mathbf{P}_n$ with C_5 independent of $p \in [1, \infty]$. In [4, Theorem 5.1], (1) and (2) were generalized for $p = 2$:

$$\|gPW\|_{L_2(-\infty, \infty)} \leq (1 + C_6 (0.9)^{2n}) |3q_{2n}|^\gamma \|PW\|_{L_2(-3|q_{2n}|, 3|q_{2n}|)},$$

$P \in \mathbf{P}_n$, $n \geq n_1$. The weight W in [4] satisfied the following conditions:

$$(3) \quad \begin{array}{l} W \text{ is nonnegative in } \mathbf{R} \text{ and measurable, with all moments} \\ \text{finite, and for some } A, B > 0, W(x) \text{ is continuous for } |x| \geq A \\ \text{and} \end{array}$$

$$\min_{A \leq |u| \leq |x|} W(u) \geq BW(x) > 0, \quad |x| \geq A.$$

In particular, the last inequality holds if $W(x)$ is even and nonincreasing for large $|x|$.

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The numbers q_t , $t > 0$, are defined as the root of smallest absolute value of the equation

$$(4) \quad W(q_t)|q_t|^t = \max_{|x| \geq A} \{W(x)|x|^t\},$$

taken as positive if both negative and positive values may be chosen. Finally, in [4] the function g satisfies the following conditions with $D = 1$:

$$(5) \quad \begin{aligned} &g: \mathbf{R} \rightarrow \mathbf{R} \text{ is measurable and for some } D > 0 \text{ and } \gamma \geq 0, \\ &|g(x)| \leq D(\max\{1, |x|\})^\gamma, \quad x \in \mathbf{R}. \end{aligned}$$

The proofs of these inequalities in [2, 3, 4, 6] depend upon estimates for the Christoffel functions, properties of the Hermite weight and convexity arguments. In this note we use Cartan's Lemma on small values of polynomials [1, p. 174] to prove inequalities such as

THEOREM A. *Let W and g be as in (3) and (5) respectively. Let $0 < \theta \leq 1$. There exist $\rho \in (0, 1)$ and n_1 depending on W , g and θ but not on n , p or P such that*

$$(6) \quad \|gPW\|_{L_p(-\infty, \infty)} \leq (1 + \rho^n)^{1/\theta} D(11|q_{2n}|)^\gamma \|PW\|_{L_p(-11|q_{2n}|, 11|q_{2n}|)},$$

$$\theta \leq p \leq \infty, P \in \mathbf{P}_n, n \geq n_1.$$

For the Hermite weight, Theorem A removes the restriction $p \geq 1$ in (1) and (2) though $11q_{2n} = 11\sqrt{2n} > 4\sqrt{n}$.

Inequalities such as the one in Theorem A are useful because they reduce problems over an infinite interval to problems on a finite interval, thereby enabling one to apply results for orthogonal polynomials on a finite interval, such as Szego's theory, to weights over an infinite interval. Further, they are essential in estimating the Christoffel functions and in investigating mean convergence of Lagrange interpolation and orthonormal expansions for weights on the whole real line.

Recently, Mhaskar and Saff [5] undertook a penetrating investigation of the L_∞ version of the inequalities in Theorem A for $g = 1$ and the weights $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 0$, using complex methods. They showed [5, Theorem 2.7] that for $P \in \mathbf{P}_n$,

$$\|PW_\alpha\|_{L_\infty(-\infty, \infty)} = \|PW_\alpha\|_{L_\infty(-a_n(\alpha), a_n(\alpha))}$$

where $a_n(\alpha) = (n/\lambda_\alpha)^{1/\alpha}$ cannot be reduced. Further they obtained an L_p inequality [5, Lemma 6.3] like that in Theorem A, for $g = 1$, with constants depending on $p \in (0, \infty]$ and implicit assumptions on the weight W [5, (6.13–6.15)]. It is unlikely that (6.13) in [5] will hold whenever (3) holds and (6.15) in [5] is not satisfied by weights like

$$W(x) = \exp(-|\log|x|| |\log|\log|x|||)$$

which satisfy (3) above. However, it seems certain that there are weights not satisfying (3) but for which (6.13–6.15) in [5] are true.

2. The general inequality. Throughout, let $\psi: \mathbf{R} \rightarrow [0, \infty)$ be an even function, monotone increasing on $(0, \infty)$, continuous in $\mathbf{R}$, with $\psi(0) = 0$. Let

$$(7) \quad \psi^*(y) = \sup_{x>0} \psi(xy)/\psi(x), \quad y \in \mathbf{R},$$

and assume this is finite in $\mathbf{R}$. It is trivially finite in $[0, 1]$. We also assume there exists $\eta > 0$ such that for $r \in [0, 1]$,

$$(8) \quad I[r] = \int_0^r \psi^*(v) v^{-1-\eta} dv < \infty.$$

THEOREM B. Let W and g be as in (3) and (5) respectively. Let ψ be as above and ψ^* be as in (7) and (8). Let $0 < \varepsilon \leq 14$. There exist C_γ and n_1 depending on $g, W, A, B, \varepsilon, D, \gamma$ but independent of ψ such that for $n \geq \max\{n_1, (\eta^{-1} + \gamma)/\varepsilon\}$ and all $P \in \mathbf{P}_n$,

$$(9) \quad \int_{-\infty}^{\infty} \psi(gPW) dx \leq \psi^*(D\xi_n^\gamma) \int_{-\xi_n}^{\xi_n} \psi(PW) dx \{1 + C_\gamma n^{-1} I[B^{-1}(0.99)^n]\}$$

where

$$(10) \quad \xi_n = 11^{1/\varepsilon} |q_{n(1+\varepsilon)}|, \quad n \geq n_1.$$

PROOF. Write $H = 11^{1/\varepsilon}$ and $\chi = |q_{n(1+\varepsilon)}|$. Let $P \in \mathbf{P}_n$, $P \neq 0$, and write

$$P(u) = c \prod_{i=1}^m (u - x_i)$$

where $m \leq n$, $|x_i| \leq H\chi$ for $1 \leq i \leq j$ and $|x_i| > H\chi$ for $j < i \leq m$. Then if $|x| \geq H\chi$, $|u| \leq \chi$ and $j < i \leq m$, we have

$$\frac{|x - x_i|}{|u - x_i|} \leq \frac{1 + |x|/(H\chi)}{1 - |u|/(H\chi)} \leq \frac{2|x|}{(H-1)\chi}.$$

Let $r = (1 - \delta)/(2e)$ where $\delta > 0$ is chosen so small that $2/(11r) < 0.99$. We see $H - 1 > r$ for $0 < \varepsilon \leq 14$. Hence,

$$(11) \quad \frac{|P(x)|}{|P(u)|} \leq \left(\frac{2|x|}{r\chi}\right)^{m-j} (2|x|)^j \left|\prod_{i=1}^j (u - x_i)\right| \leq \left(\frac{2|x|}{r\chi}\right)^m,$$

for $|x| \geq H\chi$, $|u| \leq \chi$ and $u \notin \mathcal{S}$, where $\mathcal{S}$ is open and has linear measure at most $4er\chi$. Here we have used Cartan's Lemma on small values of polynomials (see, for example, [1, p. 174]). Let $\mathcal{M} = [-\chi, \chi] \setminus (\mathcal{S} \cup (-A, A))$. Choose $\sigma = \pm 1$ so that $\sigma\chi = q_{n(1+\varepsilon)}$. By (5) and (11), and as $m \leq n$, we have for $|x| \geq H\chi > 1$, $u \in \mathcal{M}$,

$$\begin{aligned} |g(x)P(x)W(x)| &\leq \frac{D}{B} (H\chi)^\gamma \left(\frac{|x|}{H\chi}\right)^{\gamma-n\varepsilon} \left(\frac{2}{11r}\right)^n |P(u)| BW(\sigma\chi) \frac{|x|^{n(1+\varepsilon)} W(x)}{\chi^{n(1+\varepsilon)} W(\sigma\chi)} \\ &\leq \frac{D}{B} \xi_n^\gamma \left(\frac{|x|}{\xi_n}\right)^{\gamma-n\varepsilon} (0.99)^n |P(u)| W(u), \end{aligned}$$

by (3), (4) and (10). Using (7) twice, we see

$$(12) \quad \begin{aligned} & \psi(g(x)P(x)W(x)) \\ & \leq \psi^*(D\xi_n^\gamma)\psi^*\left(B^{-1}(|x|/\xi_n)^{\gamma-n\epsilon}(0.99)^n\right)\psi(P(u)W(u)), \\ & \quad |x| \geq \xi_n, u \in \mathcal{M}. \end{aligned}$$

Further, as $\mathcal{M}$ has linear measure at least $2\chi - 4\epsilon\chi - 2A = 2(\delta\chi - A)$, we have for $\chi > A/\delta$ and a certain $u \in \mathcal{M}$,

$$(13) \quad \psi(P(u)W(u)) \leq \frac{1}{2(\delta\chi - A)} \int_{\mathcal{M}} \psi(P(v)W(v)) dv.$$

Then (12) and (13) yield

$$\begin{aligned} & \int_{|x| \geq \xi_n} \psi(gPW) dx / \left\{ \psi^*(D\xi_n^\gamma) \int_{-\xi_n}^{\xi_n} \psi(PW) dx \right\} \\ & \leq \frac{1}{(\delta\chi - A)} \int_{\xi_n}^{\infty} \psi^*\left(B^{-1}\left(\frac{x}{\xi_n}\right)^{\gamma-n\epsilon}(0.99)^n\right) dx \\ & = \frac{\xi_n}{\delta H^{-1}\xi_n - A} \frac{[B^{-1}(0.99)^n]^{1/(n\epsilon-\gamma)}}{n\epsilon - \gamma} \int_0^{B^{-1}(0.99)^n} \psi^*(v) v^{-1-1/(n\epsilon-\gamma)} dv \\ & \leq C_7 n^{-1} I[B^{-1}(0.99)^n] \end{aligned}$$

for $n \geq n_1$ such that also $(n\epsilon - \gamma)^{-1} < \eta$. Here n_1 and C_7 have the properties in the statement of Theorem B. The result follows by a single application of (5) and (7). $\square$

PROOF OF THEOREM A. Let $\eta = \theta/2$. Let $\psi(t) = |t|^p$ where $\infty > p \geq \theta$. We see then $\psi^*(t) = |t|^p$ and $\psi^*(t) \leq t^\theta$, $t \in [0, 1]$, so that, for $n \geq n_2 = n_2(B)$,

$$I[B^{-1}(0.99)^n] \leq (2/\theta)(B^{-1}(0.99)^n)^{\theta/2},$$

the right side being independent of $p \geq \theta$. Substituting in (9) and (10) with $\epsilon = 1$ and taking p th roots, we obtain (6) for $\theta \leq p < \infty$. As (5) holds, and as the constants in (6) are independent of $\psi(t) = |t|^p$, we can let $p \rightarrow \infty$ to obtain the inequality for $p = \infty$ also. $\square$

REMARKS. (a) We note that $\psi(t) = |t|^p(1 + |\log|t||)^q$, $p > 0$, $q \in \mathbf{R}$, also satisfies (8), with $\psi^*(t) = \psi(t)$ if $q \geq 0$ and $\psi^*(t) = |t|^p$ if $q < 0$. However, ψ is monotone increasing throughout $(0, \infty)$ only if $p \geq |q|$.

(b) One cannot in general replace ξ_n in (9) and (10) by $\delta|q_n|$ with δ arbitrarily small [4, Lemma 5.2].

(c) Theorem B is useful in finding upper bounds for generalized Christoffel numbers

$$\lambda_n(W; \psi; \Phi) = \inf_{P \in \mathbf{P}_{n-1}} \int_{-\infty}^{\infty} \psi(PW) dx / \psi(\Phi(P))$$

where Φ is a real functional defined on all polynomials.

(d) If one weakens (3) to allow

$$\min_{A \leq |u| \leq |x|} W(u) \geq h(x)W(x), \quad |x| \geq A,$$

where h is even, positive and nonincreasing, and $\limsup_{n \rightarrow \infty} h(\xi_n)^{1/n} < \infty$, then (9) is still valid if one replaces B^{-1} by $h(\xi_n)^{-1}$. The continuity of $W(x)$ for $|x| \geq A$ may also be dropped, provided (3) and the definition (4) of q_i are suitably modified.

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