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SINGULAR FUNCTIONS AND DIVISION IN $H^\infty + C$

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ABSTRACT. In this paper it is shown that for each inner function u, there exists a singular inner function S which is divisible in $H^\infty + C$ by all positive powers of u.

Introduction. In this paper, we continue the study of division in $H^\infty + C$ begun by Guillory and Sarason. We let H^∞ denote the space of boundary functions for bounded analytic functions in the open unit disk **D** and C denote the space of continuous, complex valued functions on $\partial\mathbf{D}$. We let L^∞ denote the usual Lebesgue space with respect to Lebesgue measure. It is well known that $H^\infty + C$ is a closed subalgebra of L^∞. The space H^∞ (or $H^\infty + C$) will be identified with its analytic (or harmonic) extension to **D**.

C. Guillory and D. Sarason began the study of division in $H^\infty + C$ by determining a criterion for deciding whether an $H^\infty + C$ function is divisible by all positive powers of a unimodular $H^\infty + C$ function [**3**]. In the same paper, the question of finding, for each inner function u, a singular inner function which is divisible in $H^\infty + C$ by all positive powers of u, is posed. We shall answer this question affirmatively. The techniques used to prove this are a combination of the techniques used in [**1** and **3**]. As in [**1**], our main tools are interpolating Blaschke products and the Chang-Marshall Theorem. A sequence $\{z_n\}$ of distinct points in D is called an interpolating sequence if there exists $\delta > 0$ such that

$$\prod_{j \neq k} \left| \frac{z_k - z_j}{1 - \bar{z}_j z_k} \right| \geqslant \delta > 0, \qquad k = 1, 2, 3, \ldots.$$

It is well known [**4**, p. 199] that if a sequence of points $\{z_n\}$ of the open unit disk is an interpolating sequence, then

$$(*) \qquad \sum_{k=1}^{\infty} \left(1 - |z_k|^2\right) < \infty.$$

A Blaschke product with a zero sequence which is an interpolating sequence is called an interpolating Blaschke product.

The Chang-Marshall Theorem states that every closed subalgebra of L^∞ which contains H^∞ is generated by H^∞ and some collection of conjugates of interpolating Blaschke products.

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From the proof of the Chang-Marshall Theorem, it is easy to show that the closed subalgebra generated by H^∞ and the conjugate of one inner function is actually equal to the closed algebra generated by H^∞ and the conjugate of a single interpolating Blaschke product. We refer the reader to [**2**, Chapter IX].

The Main Theorem. In this section we prove the following theorem:

MAIN THEOREM. *For each inner function u, there exists a singular inner fucntion which is divisible in $H^\infty + C$ by all positive powers of u.*

The proof of the Main Theorem requires three lemmas. Lemmas 1 and 2 below reduce the problem to the case in which u is an interpolating Blaschke product. We then use Lemma 3 to complete the proof of the Main Theorem.

LEMMA 1. *Let u be an inner function. There exists an interpolating Blaschke product b such that if an inner function v is divisible in $H^\infty + C$ by all positive powers of b, then v is divisible in $H^\infty + C$ by all positive powers of u.*

PROOF. It follows from (the proof of) the Chang-Marshall Theorem that there exists an interpolating Blaschke product b such that the closed subalgebra of L^∞ generated by H^∞ and $\bar{u}$ is actually equal to the closed subalgebra generated by H^∞ and the conjugate of the interpolating Blaschke product b. Let v be an inner function divisible by all positive powers of b. It is easy to see that v must be divisible in $H^\infty + C$ by all positive powers of u.

The maximal ideal space of H^∞, denoted $M(H^\infty)$, is the set of nonzero complex multiplicative linear functionals on H^∞. With the weak-* topology, $M(H^\infty)$ is a compact Hausdorff space. We identify $\mathbf{D}$ with its natural image in $M(H^\infty)$.

LEMMA 2. *Let b be an interpolating Blaschke product with zero sequence $\{z_n\}$. If S is a singular inner function such that $S(z_n) \to 0$, then S is divisible by all positive powers of b.*

PROOF. For each positive integer n, let g_n be an analytic nth root of S. Thus $g_n^n = S$, $g_n \in H^\infty$ and, for each n, $g_n(z_m) \to 0$ as $m \to \infty$. Suppose $m \in M(H^\infty) \sim \mathbf{D}$ and $m(b) = 0$. By [**4**, p. 205], we have $m \in \overline{\{z_n\}}$. Hence $m(g_n) = 0$. It follows from Lemma 1 of [**1**] that $g_n\bar{b} \in H^\infty + C$. Thus $g_n^n\bar{b}^n \in H^\infty + C$ for each n and $S\bar{b}^n \in H^\infty + C$, as desired.

The techniques used to construct the singular function S are similar to those used in [**3**]. The construction will be done on the upper half-plane.

LEMMA 3. *Let $\{z_n\}$ be an interpolating Blaschke sequence. There exists a singular inner function S satisfying $S(z_n) \to 0$.*

PROOF. If $A = \{n\colon \operatorname{Re} z_n \geqslant 0\}$ is finite, then we need only consider the set $\{z_n\}$ such that $\operatorname{Re} z_n < 0$. Assume there are infinitely many z_n such that $\operatorname{Re} z_n \geqslant 0$. For those n, let $w_n = i((1 - z_n)/(1 + z_n))$. Then $\operatorname{Im} w_n > 0$ and, from (*), we have $\sum_n \operatorname{Im} w_n < \infty$. Let $\{b_n\}$ be a sequence of positive real numbers such that

$\Sigma_n b_n(\operatorname{Im} w_n) < \infty$ and $\lim_{n\to\infty} b_n = \infty$. Let $w_n' = \operatorname{Re} w_n + i b_n \operatorname{Im} w_n$ and $t_n = \operatorname{Re} w_n$. Finally, let u be the Poisson integral of the measure $\mu = \Sigma_n(\operatorname{Im} w_n')\delta_{t_n}$, that is

$$u(x, y) = \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2}\, d\mu(t) = y \sum_n \frac{b_n(\operatorname{Im} w_n)}{(x - t_n)^2 + y^2}.$$

Then

(a) $$\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} = \sum_n \frac{b_n \operatorname{Im} w_n}{1 + (t_n)^2}$$

and since $\Sigma_n b_n \operatorname{Im} w_n/(1 + t_n^2) \leqslant \Sigma_n b_n \operatorname{Im} w_n$ we have $\Sigma_n b_n \operatorname{Im} w_n/(1 + t_n^2) < \infty$.

(b) $$u(\operatorname{Re} w_m, \operatorname{Im} w_m) = \sum_n \frac{b_n(\operatorname{Im} w_n)(\operatorname{Im} w_m)}{(\operatorname{Re} w_m - t_n)^2 + (\operatorname{Im} w_m)^2}$$

so we have

$$u(\operatorname{Re} w_m, \operatorname{Im} w_m) > \frac{b_m(\operatorname{Im} w_m)^2}{(\operatorname{Re} w_m - \operatorname{Re} w_m)^2 + (\operatorname{Im} w_m)^2} = b_m.$$

Let $\tilde{u}$ be the harmonic conjugate of u, and let $S_1 = e^{-(u+i\tilde{u})}$ denote the singular inner function for the upper half-plane corresponding to μ. Then $|S_1(w_m)| = |e^{-u(w_m)}| < e^{-b_m/2}$. Hence $S_1(w_m) \to 0$ as $m \to \infty$. Letting $S_2(z) = S_1((i - z)/(i + z))$ we obtain a singular inner function such that $S_2(z_n) \to 0$ as $n \to \infty$ and $n \in A$.

Suppose now that $\{n\colon \operatorname{Re} z_n < 0\}$ is infinite. Let $w_n = i((1 + z_n)/(1 - z_n))$ for all n such that $\operatorname{Re} z_n < 0$. Again, $\operatorname{Im} w_n > 0$ and $\Sigma_n \operatorname{Im} w_n < \infty$. Repeating the process above, we obtain a singular inner function S_3 such that z_n with $\operatorname{Re} z_n < 0$ we have $S_3(z_n) \to 0$ as $n \to \infty$. If we let $S = S_2 S_3$, then S satisfies the desired conditions.

To establish the Main Theorem, let u be an inner function. Choose an interpolating Blaschke product b satisfying the conditions of Lemma 1. Use Lemma 3 to obtain a singular function S satisfying the conditions of Lemma 2. Then $\bar{b}^n S \in H^\infty + C$ for all positive integers n. By Lemma 1 we see that S is divisible by all positive powers of u.

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