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HOMOGENEITY IS NOT A WHITNEY PROPERTY

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ABSTRACT. It is proved by showing a counterexample that homogeneity is not a Whitney property.

It has been asked in [**1**, §6, p. 179] (cf. [**2**, (14.36), p. 432]) whether homogeneity of a continuum is a Whitney property. This paper contains a negative answer to this question.

We use, for brevity, concepts and definitions as in [**3**], so the paper can only be read in conjunction with [**3**].

We construct a sphere S, a Whitney map $\mu\colon C(S) \to [0, \infty)$, a number $t \in (0, \mu(S))$ and two continua belonging to $\mu^{-1}(t)$ such that $\mu^{-1}(t)$ is locally contractible at one of them, but is not at the other. This shows $\mu^{-1}(t)$ is not homogeneous, while S is.

Let ρ be the Hausdorff distance on $C(S)$ and for $K \in C(S)$, let $B_\rho(K, \delta)$ denote the δ-ball about K in $C(S)$. Let $\hat{\mu}$ be the special Whitney map for $C(S)$ as defined in [**3**, p. 277], and put $C(K, t) = C(K) \cap \hat{\mu}^{-1}(t)$. Further, denote by S^1 the unit circle, and put $S^2 = S^1 \times [-1, 1]/S^1 \times \{-1\}, S^1 \times \{1\}$ (cf. [**3**, Example 2, p. 277]).

Let A_0, A_n, X_n for $n \in \{1, 2, \ldots\}$, Z, Y and X be as in [**3**, Example 5, pp. 278–279]. Denote by A'_n, X'_n, Z', Y' and X' the images of A_n, X_n, Z, Y and X correspondingly under the symmetry with respect to the plane $z = 0$. Define $S = X \cup X'$. So S is a 2-sphere. Put $t = \hat{\mu}(A_0) = \hat{\mu}(A_n) = \hat{\mu}(A'_n)$ for all $n \in \{1, 2, \ldots\}$. We show $\hat{\mu}^{-1}(t)$ is not locally contractible at A_0. Suppose the contrary. Let $\varepsilon > 0$ be given. Then, arguing as in [**3**, p. 279], we see there exists an index i such that $C(X_i, t) \subset B_\rho(A_0, \delta) \cap \hat{\mu}^{-1}(t)$, where $B_\rho(A_0, \delta) \cap \hat{\mu}^{-1}(t)$ is contractible over $B_\rho(A_0, \varepsilon) \cap \hat{\mu}^{-1}(t)$. Let $f\colon S^2 \to \hat{\mu}^{-1}(t)$ and $g\colon C(X, t) \to S^2$ be defined as in [**3**, p. 279]. We shall now define a mapping $g^*\colon \hat{\mu}^{-1}(t) \to S^2$ so that $g^* f$ is the identity map on S^2. If $B \in \hat{\mu}^{-1}(t)$ and $B \cap \operatorname{int} Z \neq \varnothing$ or $B \cap \operatorname{int} Z' \neq \varnothing$, then $g^*(B) = S^1 \times \{1\}$. For $n \in \{1, 2, \ldots, i\}$ let $g^*(A_n) = g^*(A'_n) = S^1 \times \{1\}$; for $n \in \{i+1, i+2, \ldots\}$ let $g^*(A_n) = g^*(A'_n) = S^1 \times \{-1\}$ and let $g^*(A_0) = S^1 \times \{-1\}$. Let $p^*\colon Y \cup Y' \to A_0$ be the map obtained by projecting (either downward or upward) along great circles in the spheres containing either A_n and A_{n+1} or A'_n and A'_{n+1} for $n \in \{1, 2, \ldots\}$, and let $q^*\colon Y \cup Y' \to [-1, 1]$ be the projection onto the z-axis. For $B \in \hat{\mu}^{-1}(t) \cap C(Y \cup Y')$, $A_j \neq B \neq A'_j$ for all j, let $e^{2\pi i \theta(B)}$ and $r(B)$ be the mid-points of $p^*(B)$ and $q^*(B)$ respectively. If $r(B) \geqslant 1/2^{i-1}$ or $r(B) \leqslant -1/2^{i-1}$, let

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$g^*(B) = S^1 \times \{1\}$ and if $-1/2^i < r(B) < 1/2^i$, let $g^*(B) = S^1 \times \{-1\}$. If $1/2^i < r(B) < 1/2^{i-1}$, let $g^*(B) = (e^{2\pi i\theta(B)}, 2^{i+1}(r(B)-3))$ and if $-1/2^i > r(B) > -1/2^{i-1}$, let $g^*(B) = (e^{2\pi i\theta(B)}, 2^{i+1}(r(B) + 3))$. Observe that $g^*|C(X, t) = g$. Using arguments as in [**3**, p. 280] we see that g^* is continuous and $g^*(f(x)) = x$ for all $x \in S^2$.

We have a contradiction since f is homotopic to a constant mapping, while g^*f is not.

Now let $A = f(s, 0)$, where $s \in S^1$. We show that $\hat{\mu}^{-1}(t)$ is locally contractible at A. Indeed, the family of sets of the form $C(\operatorname{cl} V_{1/n}(A), t)$, where $n \in \{1, 2, \dots\}$, is a local base of A in $\hat{\mu}^{-1}(t)$, but for sufficiently large n the sets $\operatorname{cl} V_{1/n}(A)$ are continua satisfying assumptions of Lemma 6 of [**3**, p. 280]; hence we conclude that the elements of the local base are contractible, and so we are done.

The referee has raised the following four interesting questions connected with the topic of this paper.

1. Does there exist a homogeneous continuum X such that for every Whitney map μ of X, there exists a t with $\mu^{-1}(t)$ nonhomogeneous?

2. If X is a homogeneous continuum and μ is a Whitney map of X, are most (e.g., a dense G_δ-set) Whitney levels of μ homogeneous?

3. If X is a homogeneous continuum and μ is a Whitney map of X, must each $\mu^{-1}(t)$ contain a dense G_δ orbit?

4. Is there a characterization in terms of other topological properties of homogeneous continua where every Whitney level of every Whitney map is homogeneous? Such continua do exist.

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