DIFFERENTIAL BASIS AND p -BASIS OF A REGULAR LOCAL RING

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ABSTRACT. We prove that any differential basis of a regular local ring R of characteristic $p > 0$ over R^p is a p -basis of R over R^p . This result gives a characterization of a regular local ring R which has a p -basis over R^p .

1. Introduction. Let R be a local ring of characteristic $p > 0$ and let R' be a sublocal ring of R . We denote the differential module of R over R' by $\Omega_{R'}(R)$. If $\Omega_{R'}(R)$ is a free R -module and if $\{da_i\}_{a_i \in \Lambda}$ is a basis of the R -module $\Omega_{R'}(R)$, $\Lambda = \{a_i\}_{i \in I}$ is called a differential basis of R over R' . If R has a p -basis Γ over R' , Γ is also a differential basis of R over R' (see [6, §38]). Conversely, any differential basis is a p -basis under the assumptions that $R' \supset R^p$ and R is a finite R' -module (see [6, §38, Proposition]). The purpose of this paper is to prove that any differential basis of a regular local ring R over R^p is a p -basis of R over R^p . This result gives a characterization of a regular local ring R which has a p -basis over R^p (cf. [5, Theorem 3.1]). This fact has not been noticed, to the best of the authors' knowledge, in any literature yet.

2. Notations and preliminaries. In this paper, p is always a prime number and all rings are commutative with identity. A ring is called a local ring if it is noetherian and has only one maximal ideal. Let S be a ring of characteristic p and S^p denote the subring $\{x^p | x \in S\}$. Let S' be a subring of S . A subset Γ of S is said to be p -independent over S' , if the monomials $b_1^{e_1} \cdots b_n^{e_n}$ where $b_1, \dots, b_n$ are distinct elements of Γ and $0 \leq e_i \leq p-1$, are linearly independent over $S^p[S']$. Γ is called a p -basis of S over S' if it is p -independent over S' and $S^p[S', \Gamma] = S$. We denote the differential module of S over S' by $\Omega_{S'}(S)$ and the differentiation of S over S' which define $\Omega_{S'}(S)$ by d . For the definition and elementary properties, refer to [6].

From now on throughout this paper, R will denote a local ring of characteristic p , M the maximal ideal of R , k the residue field of R , and (if R is a domain) K the quotient field of R . We denote the Krull dimension of R by $\dim R$ and we put $\dim R = r$. We set $M^{(p)} = \{m^p | m \in M\}$. Since $M \cap R^p = M^{(p)}$, the natural map $R^p/M^{(p)} \rightarrow R/M = k$ is injective and its image is equal to $(R/M)^p = k^p = \{\alpha^p | \alpha \in k\}$. In view of the above injection, the residue field $R^p/M^{(p)}$ of R^p can be

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identified with the subfield k^p of k . R' will denote an intermediate local ring between R and R^p , M' the maximal ideal, k' the residue field. It is clear that R dominates R' , that is, $M \cap R' = M'$. Since we may identify the residue field k' of R' with the corresponding subfield of k , we may assume that $k^p \subset k' \subset k$. For any subset A of R , we denote by $\bar{A}$ the set of residue classes of the elements of A modulo M . When we say " $\bar{A}$ is a p -basis" we tacitly assume that the natural mapping $A \rightarrow \bar{A}$ maps A injectively to $\bar{A}$.

3. Lemma.

LEMMA 1 [6, §38, Proposition]. *Let (R, M) be a noetherian local ring of characteristic p and R' be a subring of R containing R^p such that R is finite over R' . Put $M' = M \cap R'$, $k = R/M$ and $k' = R'/M'$. If $\Omega_{R'}(R)$ is a free R -module with $dx_1, \dots, dx_r$ ($x_i \in R$) as a basis, then $x_1, \dots, x_r$ form a p -basis of R over R' .*

REMARK. Let S be a ring of characteristic p and S' a subring of S containing S^p and let $\{x_1, \dots, x_n\}$ be a subset of S . If $\{dx_1, \dots, dx_n\}$ is S -free in $\Omega_{S'}(S)$, $\{x_1, \dots, x_n\}$ is p -independent over S' . Therefore, a differential basis of S over S' is p -independent over S' (cf. [6, §38]).

LEMMA 2. *Let S be a ring of characteristic p . Let Λ be a differential basis of S over S^p and let Λ_1 be a subset of Λ . Then Λ_1 is a differential basis of $S^p[\Lambda_1]$ over S^p and $\Lambda_2 = \Lambda - \Lambda_1$ is a differential basis of S over $S^p[\Lambda_1]$.*

PROOF. It is easy to prove Lemma 2.

LEMMA 3. *Let (R, M, k) be a local ring of characteristic p . If Λ is a differential basis of R over R^p , there is a subset A of Λ such that $\bar{A}$ is a p -basis of k over k^p .*

PROOF. We have a natural surjective homomorphism

$$\Omega_{R^p}(R) \rightarrow \Omega_{k^p}(k),$$

which maps da to $d\bar{a}$ where $\bar{a}$ is the residue class of $a \in \Lambda$ modulo M . Since $\Omega_{R^p}(R)$ is generated by $\{da | a \in \Lambda\}$, $\Omega_{k^p}(k)$ has a basis which is a subset of $\{d\bar{a} | a \in \Lambda\}$. Let A be a subset of Λ such that $\{d\bar{a} | a \in \Lambda\}$ is a basis of $\Omega_{k^p}(k)$ and such that the natural map $A \rightarrow k$ is one-to-one. Then $\bar{A} = \{\bar{a} | a \in A\}$ is a p -basis of k/k^p .

LEMMA 4. *Let (R, M, k) be a local ring of characteristic p and let A be a subset of R such that $\bar{A}$ is a p -basis of k over k^p . Then we have*

$$M/M^2 \cong \Omega_{R^p[A]}(R) \otimes_R k.$$

PROOF. $R^p[A]$ is a local ring with the maximal ideal $M_A = M^{(p)}R^p[A]$ and the residue field k by Lemma 2.1 and Lemma 2.3 of [5]. From the exact sequence of Theorem 1 of [1], we have $M/M^2 \cong \Omega_{R^p[A]}(R) \otimes_R k$.

LEMMA 5. *Let (R, M, k) be a local ring of characteristic p , Λ a differential basis of R over R^p and A be a subset of Λ such that $\bar{A}$ is a p -basis of k over k^p . Then we have $|\Lambda - A| = \text{rank}_k M/M^2$.*

PROOF. Since Λ is a differential basis of R over R^p , $\Lambda - A$ is a differential basis of R over $R^p[A]$ by Lemma 2. That is, we have $|\Lambda - A| = \text{rank}_R \Omega_{R^p[A]}(R)$. On the other hand, we have $M/M^2 \cong \Omega_{R^p[A]}(R) \otimes_R k$ by Lemma 4 and hence $\text{rank}_R \Omega_{R^p[A]}(R) = \text{rank}_k M/M^2$. Then we have $|\Lambda - A| = \text{rank}_k M/M^2$.

LEMMA 6. Let (R, M, k) be a regular local ring of characteristic p and let A be a subset of R such that $\bar{A}$ is a p -basis of k over k^p . Set $\dim R = r$. Let $\{x_1, \dots, x_r\}$ be a minimal system of generators for M . Then $R^p[A][x_1, \dots, x_r]$ is a regular local ring and $\{x_1, \dots, x_r\}$ is p -independent over $R^p[A]$.

PROOF. $R^p[A]$ is a regular local ring by Lemma 2.4 and Lemma 2.5 of [5]. Set $R_r = R^p[A][x_1, \dots, x_r]$. Since $\dim R_r = r$ and the maximal ideal of R_r is generated by $\{x_1, \dots, x_r\}$, R_r is a regular local ring. Let M_r be the maximal ideal of R_r and $M_A = M^{(p)}R^p[A]$ be the maximal ideal of $R^p[A]$. We denote the M_r -adic completion and M_A -adic completion of R_r and $R^p[A]$ by $\hat{R}_r$ and $\widehat{R^p[A]}$, respectively. Then we have the following commutative diagram:

$$\begin{array}{ccc} R_r & \rightarrow & \hat{R}_r = k[[X_1, \dots, X_r]] \\ \uparrow & x_i \mapsto X_i & \uparrow \\ R^p[A] & \rightarrow & \widehat{R^p[A]} = k[[X_1^p, \dots, X_r^p]]. \end{array}$$

Since $\{X_k\}_{k=1, \dots, r}$ is p -independent over $k[[X_1^p, \dots, X_r^p]]$, $\{x_i\}_{i=1, \dots, r}$ must be p -independent over $R^p[A]$.

4. Theorem.

THEOREM. Let R be a regular local ring of characteristic p . Then any differential basis of R over R^p is a p -basis of R over R^p . Conversely, any p -basis of R over R^p is a differential basis of R over R^p .

PROOF. Let Λ be a differential basis of R over R^p . Then there is a subset A of Λ such that $\bar{A}$ is a p -basis of k over k^p by Lemma 3. Set $B = \Lambda - A$. Then $\Omega_{R^p[A]}(R)$ is a free R -module with a basis $\{db|b \in B\}$ by Lemma 2. By Lemma 5, we have $|B| = \dim R = r$. Since $\Omega_{R^p[A]}(R)$ is a free R -module of rank r , $\Omega_{K^p(A)}(K)$ is also a free K -module of rank r . Then we have $[K: K^p(A)] = p^r$. Let $\{x_1, \dots, x_r\}$ be a minimal system of generators for the maximal ideal M of R . By Lemma 6, we have $[K^p(A)(x_1, \dots, x_r): K^p(A)] = p^r$. Therefore, we get $K = K^p(A)(x_1, \dots, x_r)$. By Lemma 6, $R^p[A][x_1, \dots, x_r]$ is a regular local ring. That is, $R = R^p[A][x_1, \dots, x_r]$ and $A \cup \{x_1, \dots, x_r\}$ is a p -basis of R over R^p . Since R is a finite $R^p[A]$ -module, B is a p -basis of R over $R^p[A]$ by Lemma 1. Then $A \cup B = \Lambda$ is a p -basis of R over R^p . Conversely, any p -basis of R over R^p is a differential basis of R over R^p (cf. §38 of [6]).

COROLLARY 1. Let R be a regular local ring of characteristic p . Then the following two conditions are equivalent.

- (1) $\Omega_{R^p}(R)$ is a free R -module.
- (2) R has a p -basis over R^p .

COROLLARY 2. Let L be a field of characteristic p and $S = L[X_1, \dots, X_n]$ be a polynomial ring over L . Let P and Q be prime ideals of S such that $P \subset Q$. Set $R = (S/P)_Q$. Then the following three conditions are equivalent.

- (1) R is a regular local ring.
- (2) R has a differential basis over R^p .
- (3) R has a p -basis over R^p .

Furthermore, any differential basis of R over R^p is also a p -basis of R over R^p .

PROOF. The equivalence of (1), (2) and (3) follows from [3] and [5].

COROLLARY 3. Let R be a regular local ring of characteristic p . If $\Omega_{R^p}(R)$ is a free R -module, then R is excellent.

PROOF. If $\Omega_{R^p}(R)$ is a free R -module, then R has a p -basis over R^p by Theorem. The excellence of R follows from Theorem of [4].

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