ON INTEGRAL INEQUALITIES ASSOCIATED WITH A LINEAR OPERATOR EQUATION

M. B. SUBRAHMANYAM

ABSTRACT. In this paper we apply control theoretic concepts to formulate and solve a generalized problem for the determination of best possible constants in integral inequalities. We investigate the problem of existence of functions for which the best constant is attained, and also the conditions satisfied by these functions.

I. Introduction. In this paper we apply control theoretic ideas to a linear operator equation of the form $w = Du$ to determine the best possible constants in integral inequalities involving w and u . The generalized approach of this paper encompasses all the previous cases considered [1-4]. On the other hand, it gives rise to integral inequalities in a number of new settings.

Consider the continuous linear operator

$$(1.1) \quad D: L_p^m(A, \mu) \rightarrow L_r^n(B, \nu), \quad p, r > 1,$$

where $A \subseteq R^k$ and $B \subseteq R^l$ are subsets. $L_p^m(A, \mu)$ denotes the Banach space of m -dimensional measurable functions on A with the norm

$$(1.2) \quad \|u\|_p = \left[\int_A |u(t)|^p d\mu(t) \right]^{1/p},$$

where $|\cdot|$ denotes the finite-dimensional Euclidean norm, and μ a positive measure on A . Similar remarks apply to $L_r^n(B, \nu)$.

Let

$$(1.3) \quad w = Du.$$

Our problem is to select u such that

$$(1.4) \quad F(w, u) = \frac{\int_A \phi_1(u, t) d\mu(t)}{[\int_B \phi_2(w, x) d\nu(x)]^\alpha}$$

is minimized, where $\alpha > 0$, and ϕ_1, ϕ_2 are real-valued functions. Note that if the minimum value of (1.4) is $\beta > 0$, we have the natural inequality

$$(1.5) \quad \int_A \phi_1(u, t) d\mu(t) \geq \beta \left[\int_B \phi_2(w, x) d\nu(x) \right]^\alpha,$$

where β is the best possible constant.

Received by the editors January 27, 1983.

1980 *Mathematics Subject Classification.* Primary 49A29; Secondary 26D15.

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II. Existence of best possible constant. We make the following assumptions:

(i) ϕ_i is continuous in its arguments for $i = 1, 2$. Also ϕ_1 is convex in u for each t .

(ii) Admissible u are such that $|\int_A \phi_1(u, t) d\mu(t)| < \infty$.

(iii) $\phi_2(w, x) \geq 0$ for any w which satisfies (1.3) for some admissible u .

(iv) $\nabla_u \phi_1 \in L_q^m(A, \mu)$ for all $u \in L_p^m(A, \mu)$, $1/p + 1/q = 1$.

(v) For each $K < \infty$ there exists a ν -integrable g_K on B such that for each $\|u\|_p \leq K$,

$$|\phi_2(w, x)| \leq g_K \quad \text{a.e. } [\nu] \text{ on } B.$$

(vi) $D: L_p^m(A, \mu) \rightarrow L_r^n(B, \nu)$ maps weakly convergent sequences into strongly convergent ones.

(vii) There exists $k > 0$ such that

$$\phi_1(cu, t) = c^k \phi_1(u, t), \quad \phi_2(cw, x) = c^{k/\alpha} \phi_2(w, x).$$

(viii) There is an admissible u such that $0 < \int_B \phi_2(w, x) d\nu(x) < \infty$.

THEOREM 1.1. Consider (1.3) and (1.4) along with assumptions (i)–(viii). Then there exists an admissible u that minimizes (1.4).

PROOF. By the assumptions it is sufficient to exhibit a minimizing u among all u for which $\int_B \phi_2(w, x) d\nu(x) = M > 0$. Let $\inf_u \int_A \phi_1(u, t) d\mu(t) = J$ subject to $\int_B \phi_2(w, x) d\nu(x) = M$. Let $\{u_n\}$ be such that $\lim_{n \rightarrow \infty} \int_A \phi_1(u_n, t) d\mu(t) = J$ with $\int_B \phi_2(w_n, x) d\nu(x) = M$ for all n . Then a subsequence, still denoted by $\{u_n\}$, converges to some $u_0 \in L_p^m(A, \mu)$ weakly. By assumption (vi), w_n converges to w_0 , and by assumption (v) and the Lebesgue dominated convergence theorem, we have $\int_B \phi_2(w, x) d\nu(x) = M$.

By assumption (i),

$$\phi_1(u_n(t), t) - \phi_1(u_0(t), t) \geq \nabla_u \phi_1(u_0(t), t)(u_n(t) - u_0(t))$$

for almost all $t \in A$. Integrating both sides on A and letting $n \rightarrow \infty$, we get, by assumptions (iv) and (v),

$$\int_A \phi_1(u_0, t) d\mu(t) \leq \liminf_{n \rightarrow \infty} \int_A \phi_1(u_n, t) d\mu(t) = J.$$

COROLLARY 1.1. Let $A = B$, $\mu = \nu$, and

$$(2.1) \quad F(w, u) = \frac{\int_A \phi_1(w, u, t) d\mu(t)}{[\int_A \phi_2(w, t) d\mu(t)]^\alpha},$$

where ϕ_1 is convex in both w and u for each t and $\nabla_w \phi_1 \in L_s^n(A, \mu)$, $1/r + 1/s = 1$. Also assume that $\phi_1(cw, cu, t) = c^k \phi_1(w, u, t)$. Then, under the same assumptions of Theorem 1.1, there exists an admissible control u_0 minimizing $F(w, u)$.

III. Characterization of u_0 . In this section we derive an equation that u_0 needs to satisfy. This is useful in synthesizing u_0 in particular cases. We consider equations (1.3) and (1.4) under the following assumptions.

(i) Admissible u are such that $|\int_A \phi_1(u, t) d\mu(t)| < \infty$.

(ii) $D: L_p^m(A, \mu) \rightarrow L_r^n(B, \nu)$ possesses an adjoint $D^*: L_s^n(B, \nu) \rightarrow L_q^m(A, \mu)$ ($1/p + 1/q = 1$, $1/r + 1/s = 1$) such that for all $f_1 \in L_s^n(B, \nu)$ and $f_2 \in L_p^m(A, \mu)$, we have

$$(3.1) \quad \int_B (f_1, Df_2) d\nu = \int_A (D^* f_1, f_2) d\mu.$$

Here $(\cdot, \cdot)$ denotes the standard inner product.

(iii) (w_0, u_0) minimizes (1.4) $\Rightarrow 0 < \int_A \phi_1(u, t) d\mu(t) < \infty$, $0 < \int_B \phi_2(w, x) d\nu(x) < \infty$.

(iv) Let ϕ_1 be continuously differentiable in u and measurable in t , and ϕ_2 be continuously differentiable in w and measurable in x .

(v) $\nabla_u \phi_1 \in L_q^m(A, \mu)$ for all $u \in L_p^m(A, \mu)$, $1/p + 1/q = 1$ and $\nabla_w \phi_2 \in L_s^n(B, \nu)$ whenever $w \in L_r^n(B, \nu)$, $1/r + 1/s = 1$.

(vi) There exist δ , k_1 , $k_2 > 0$ depending on (w, u) such that for all $\|h\|_r \leq \delta$ and $\|\bar{u}\|_p \leq \delta$, we have

$$(3.2) \quad \begin{aligned} \|\nabla_u \phi_1(u + \bar{u}, t) - \nabla_u \phi_1(u, t)\|_q &\leq k_1 \|\bar{u}\|_p, \\ \|\nabla_w \phi_2(w + h, x) - \nabla_w \phi_2(w, x)\|_s &\leq k_2 \|h\|_r. \end{aligned}$$

THEOREM 3.1. Consider (1.3) and (1.4) along with the above assumptions. If $(w_0, u_0) \in L_r^n(B, \nu) \times L_p^m(A, \mu)$ minimizes (1.4), then

$$(3.3) \quad \nabla_u \phi_1(u_0, t) - \alpha \lambda D^*(\nabla_w \phi_2(w_0, x)) = 0 \quad \text{a.e. } [\mu] \text{ on } A,$$

where

$$(3.4) \quad \lambda = \frac{\int_A \phi_1(u_0, t) d\mu(t)}{\int_B \phi_w(w_0, x) d\nu(x)}.$$

PROOF. Let $E = L_r^n(B, \nu) \times L_p^m(A, \mu)$. We have an equality constraint in E given by (1.1). Associated with this is the cone K_1 of tangent directions. Associated with the functional (1.4) is the cone K_0 of directions of decrease. For the definitions of various terms, see [5]. If H is a cone in E , then the dual cone $H^* = \{g \in E^* | g(\tilde{p}) \geq 0 \text{ for all } \tilde{p} \in H\}$. The Dubovitskii-Milyutin theorem states that there exist $g_0 \in K_0^*$ and $g_1 \in K_1^*$, not both zero, such that for all $(w, u) \in E$, $g_0(w, u) + g_1(w, u) = 0$.

A. Cone of directions of decrease. By assumption (iii) we can minimize the alternative cost functional

$$(3.5) \quad G(w, u) = \ln \int_A \phi_1(u, t) d\mu(t) - \alpha \ln \int_B \phi_2(w, x) d\nu(x).$$

By assumptions (iv)–(vi), the Fréchet derivative of G at (w_0, u_0) is given by

$$(3.6) \quad \begin{aligned} G'(w_0, u_0)(w, u) &= \frac{\int_A (\nabla_u \phi_1(u_0, t), u) d\mu(t)}{\int_A \phi_1(u_0, t) d\mu(t)} \\ &\quad - \alpha \frac{\int_B (\nabla_w \phi_2(w_0, x), w) d\nu(x)}{\int_B \phi_2(w_0, x) d\nu(x)}. \end{aligned}$$

By Theorem 7.5 of [5], (w, u) lies in the cone of K_0 directions of decrease iff $G'(w_0, u_0)(w, u) < 0$ or, from (3.6), iff

$$(3.7) \quad \int_A (\nabla_u \phi_1(u_0, t), u) d\mu(t) - \alpha \lambda \int_B (\nabla_w \phi_2(w_0, x), w) d\nu(x) < 0,$$

where λ is given by (3.4). If $K_0 \neq \emptyset$, then by Theorem 10.5 of [5], for any $g_0 \in K_0^*$,

$$(3.8) \quad g_0(w, u) = -\lambda_0 \left\{ \int_A (\nabla_u \phi_1(u_0, t), u) d\mu(t) - \alpha \lambda \int_B (\nabla_w \phi_2(w_0, x), w) d\nu(x) \right\}, \quad \lambda_0 \geq 0.$$

B. Cone of tangent directions. To find the tangent directions at (w_0, u_0) in E , we apply the results of Lecture 9 of [5]. Let

$$(3.9) \quad Q = \{(w, u) \in E \mid w = Du\}$$

and

$$(3.10) \quad P(w, u) = w - Du,$$

which maps E into $L_r^n(B, \nu)$. Also

$$(3.11) \quad P'(w_0, u_0)(w, u) = w - Du.$$

Clearly $P'(w_0, u_0)$ is onto $L_r^n(B, \nu)$ (just let $u = 0$). By Theorem 9.1 of [5], the set K_1 of tangent directions in E at (w_0, u_0) is given by

$$K_1 = \{(w, u) \in E \mid P'(w_0, u_0)(w, u) = 0\} = \{(w, u) \in E \mid w = Du\}.$$

Thus K_1 is a subspace and, if $g_1 \in K_1^*$, then $g_1(w, u) = 0$ for all $(w, u) \in K_1$.

C. Application of the Dubovitskii-Milyutin theorem. Let u be arbitrary and $w = Du$. Then for these pairs of (w, u) we get (since $g_1(w, u) + g_2(w, u) = 0$, $(w, u) \in E$, $(g_1, g_2) \neq 0$)

$$(3.12) \quad \int_A (\nabla_u \phi_1(u_0, t), u) d\mu(t) - \alpha \lambda \int_B (\nabla_w \phi_2(w_0, x), w) d\nu(x) = 0$$

for all u . So

$$(3.13) \quad \nabla_u \phi_1(u_0, t) - \alpha \lambda D^*(\nabla_w \phi_2(w_0, x)) = 0 \quad \text{a.e. } [\mu] \text{ on } A.$$

D. Exceptional case. If $K_0 = \emptyset$ we have

$$(3.14) \quad \int_A (\nabla_u \phi_1(u_0, t), u) d\mu(t) - \alpha \lambda \int_B (\nabla_w \phi_2(w_0, x), w) d\nu(x) = 0$$

for all $(w, u) \in E$. Thus (3.3) is satisfied in this case also.

COROLLARY 3.1. Let $A = B$, $\mu = \nu$ and

$$(3.15) \quad F(w, u) = \frac{\int_A \phi_1(w, u, t) d\mu(t)}{[\int_A \phi_2(w, t) d\mu(t)]^\alpha}.$$

In addition to assumptions (i)–(vi) of this section assume the following.

(a) ϕ_1 is continuously differentiable in (w, u) and measurable in t . Also $\nabla_w \phi_1 \in L_s^n(A, \mu)$ whenever $w \in L_r^n(A, \mu)$, $1/r + 1/s = 1$.

(b) Replace the first inequality in (3.2) by

$$(3.16) \quad \begin{aligned} & \|\nabla_u \phi_1(w + h, u + \bar{u}, t) - \nabla_u \phi_1(w, u, t)\|_q \\ & + \|\nabla_w \phi_1(w + h, u + \bar{u}, t) - \nabla_w \phi_1(w, u, t)\|_s \\ & \leq k_1[\|h\|_r + \|\bar{u}\|_p]. \end{aligned}$$

If (w_0, u_0) minimizes (3.15), then

$$(3.17) \quad \nabla_u \phi_1(w_0, u_0, t) + D^*(\nabla_w \phi_1(w_0, u_0, t) - \alpha \lambda \nabla_w \phi_2(w_0, t)) = 0$$

a.e. $[\mu]$ on A , where

$$(3.18) \quad \lambda = \frac{\int_A \phi_1(w_0, u_0, t) d\mu(t)}{\int_A \phi_2(w_0, t) d\mu(t)}.$$

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DEPARTMENT OF ELECTRICAL ENGINEERING, UNIVERSITY OF MISSOURI-COLUMBIA,
COLUMBIA, MISSOURI 65211