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# NONCONTRACTIVE UNIFORMLY LIPSHITZIAN SEMIGROUPS IN HILBERT SPACE

DARYL TINGLEY[1]

ABSTRACT. It is shown that any $k$-Lipshitzian, $k < \pi/2$, noncontractive commutative semigroup acting on a closed bounded convex set in Hilbert space has a common fixed point.

**1. Introduction.** Let $\mathcal{F} = \{f_\alpha | \alpha \in A\}$ be a semigroup of mappings of a metric space $(M, d)$ into itself. The semigroup $\mathcal{F}$ is said to have a fixed point if there exists $x_0 \in M$ with $f_\alpha(x_0) = x_0$ for all $\alpha \in A$, and $\mathcal{F}$ is said to be uniformly $k$-Lipshitzian if for each $x, y \in M$ and each $\alpha \in A$,

$$d(f_\alpha(x), f_\alpha(y)) \leq kd(x, y).$$

$\mathcal{F}$ is said to be left reversible if every two right ideals of $\mathcal{F}$ have a nonempty intersection (i.e. for $f, g \in \mathcal{F}$, $f\mathcal{F} \cap g\mathcal{F} \neq \emptyset$). Commutative semigroups, and in particular $\{f^n\colon n = 0, 1, \ldots\}$ for some function $f$, are left reversible.

In [**4**] Goebel, Kirk, and Thele showed that if $X$ is a Banach space with $\delta(1) > 0$ (where $\delta$ is the modulus of convexity function) then there is a constant $k_0' > 1$ such that any left reversible uniformly $k$-Lipshitzian semigroup $\mathcal{F}$, $k < k_0'$, acting on a closed bounded convex set $K$ in $X$ has a fixed point. Clearly there is a maximum choice of $k_0'$ which we call $k_0$. In [**4**] it was shown that for Hilbert space ($\mathcal{H}$), $\sqrt{5}/2 \leq k_0 \leq 2$. Downing and Ray [**3**] improved this estimate of $k_0$, for Hilbert space, by showing that $\sqrt{2} \leq k_0$ while in [**1**], Baillon has an example (presented here in §2) which shows that $k_0 \leq \pi/2$. (Lim [**6** and **7**] has improved the results of [**4**] for the $L^p$ spaces.)

In this note we show that under the additional assumption that $\mathcal{F}$ be a noncontractive (i.e. $\|x - y\| \leq \|f(x) - f(y)\|$ for each $f \in \mathcal{F}$ and all $x, y \in K$) uniformly $(\pi/2 - \delta)$-Lipshitzian ($\delta > 0$) commutative semigroup, then $\mathcal{F}$ has a fixed point. The example of Baillon, showing that $k_0 \leq \pi/2$, is noncontractive (this is proven in §2 of this paper), hence $\pi/2$ is the exact value of $k_0$ in the case of noncontractive commutative semigroups. We note that the example in [**4**] showing that $k_0 \leq 2$ is also noncontractive.

**2.** In this section we present Baillon's example [**1**] of a noncontractive uniformly $\pi/2$-Lipshitzian semigroup of mappings of a closed bounded convex subset of $l^2$ which contains no fixed point.

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Let $S$ be the shift operator. That is, $S((x_1, x_2, \ldots)) = (0, x_1, x_2, \ldots)$. Let $e_1 = (1, 0, 0, \ldots)$ and $K = \{x \in l^2 \colon x = (x_1, x_2, \ldots), \|x\| \le 1, x_i \ge 0, i = 1, 2, \ldots\}$. Define $f$ by

$$f(x) = \cos\left(\frac{\pi}{2}\|x\|\right) e_1 + \sin\left(\frac{\pi}{2}\|x\|\right) \frac{S(x)}{\|x\|}.$$

Several properties of $f$ are immediate,

$$\|f(x)\| = 1 \quad \text{for all } x \in K. \tag{2.1}$$

$$\text{If } \|x\| = 1 \quad \text{then } f(x) = S(x). \tag{2.2}$$

$$f^n(x) = S^{n-1} f(x) \quad \text{(from 2.1 and 2.2)}. \tag{2.3}$$

If $f(x) = x (x \in K)$ then $x = f^2(x) = Sf(x) = S(x)$. Since $S(x) = x$ implies that $x = 0$, and $f(0) = (1, 0, \ldots)$, $f$ has no fixed points. Once it is estabished that

$$\|x - y\| \le \|f(x) = f(y)\| < \pi/2\|x - y\|, \qquad x, y \in K, \tag{2.4}$$

the semigroup $\mathcal{F} = \{f^n \mid n = 1, 2, \ldots\}$ is readily seen to be noncontractive uniformly $\pi/2$-Lipshitzian with no fixed points. Note that the second inequality of 2.4 is strict, so $\pi/2$ is a strict Lipshitz constant for $\mathcal{F}$.

The fact that $\mathcal{F}$ is noncontractive has not been mentioned in the literature, we believe, and as it shows that the constant $\pi/2$ in our main theorem is the "best possible", we present a brief proof (due to the referee).

PROOF THAT $\mathcal{F}$ IS NONCONTRACTIVE. We readily compute that

$$\begin{aligned}
\|f(x) - f(y)\|^2 - \|x - y\|^2 &= \left(\cos\left(\frac{\pi}{2}\|x\|\right) - \cos\left(\frac{\pi}{2}\|y\|\right)\right)^2 \\
&\quad + \left\|\frac{x}{\|x\|}\sin\left(\frac{\pi}{2}\|x\|\right) - \frac{y}{\|y\|}\sin\left(\frac{\pi}{2}\|y\|\right)\right\|^2 - \|x - y\|^2 \\
&= 2 - 2\cos\left(\frac{\pi}{2}\|x\|\right)\cos\left(\frac{\pi}{2}\|y\|\right) - (\|x\|^2 + \|y\|^2) \\
&\quad + 2\left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle \left\{\|x\|\,\|y\| - \sin\left(\frac{\pi}{2}\|x\|\right)\sin\left(\frac{\pi}{2}\|y\|\right)\right\}.
\end{aligned}$$

The well-known inequality $\sin t \ge 2t/\pi$ (for $0 \le t \le \pi/2$) shows that the expression in curly brackets is less than or equal to 0, thus replacing $\langle x/\|x\|, y/\|y\|\rangle$ by one, we get

$$\begin{aligned}
\|f(x) - f(y)\|^2 - \|x - y\|^2 &\ge 2 - 2\cos\frac{\pi}{2}(\|x\| - \|y\|) - (\|x\| - \|y\|)^2 \\
&= 4\sin^2\frac{\pi}{4}(\|x\| - \|y\|) - (\|x\| - \|y\|)^2.
\end{aligned}$$

A final appeal to $\sin t \ge 2t/\pi$ shows that this is nonnegative. □

**3.** In this section $\mathcal{H}$ shall denote a Hilbert space, $K$ a closed bounded convex set of $\mathcal{H}$, $S$ a subset of $K$, $f \colon K \to K$ a noncontractive function, and $\mathcal{F} \colon K \to K$ a noncontractive commutative semigroup. We show that if $\mathcal{F}$ is uniformly $(\pi/2 - \delta)$-Lipshitzian, $\delta > 0$, then $\mathcal{F}$ has a fixed point.

Let $S \subseteq K$. Then $S$ is bounded, as $K$ is. For $x \in \mathcal{H}$ define

$$r(S, x) = \sup\{\|x - s\| \colon s \in S\}, \tag{3.1}$$

$$r(S) = \inf\{r(S,x)\colon\ x \in \mathcal{H}\}, \tag{3.2}$$

and let $c(S)$ be the unique point of $\mathcal{H}$ such that $r(S, c(S)) = r(S)$. Equivalently, $c(S)$ is the unique point of $\mathcal{H}$ such that $\overline{B}(c(S), r(S)) \supseteq S$ (where $\overline{B}(x,r)$ denotes the closed ball about $x$ of radius $r$). The point $c(S)$ is called the Chebyshev center of $S$. When no confusion arises, let $r(x) = r(S,x)$, $r = r(S)$ and $c = c(S)$. It is well known and easily shown (cf. [**5**]) that $c(S)$ lies in the closed convex hull of $S$ (denoted $\overline{\text{co}}(S)$) and hence is in $K$.

LEMMA 3.1. *For $x \in \mathcal{H}$, $r^2 + \|c-x\|^2 \le r^2(x)$.*

PROOF. To simplify notation, assume that $c = 0$. Then $\|y\| \le r$ for all $y \in S$. For a contadiction, suppose that

$$\varepsilon = \frac{r^2 + \|x\|^2 - r^2(x)}{2\|x\|} > 0$$

and let $z = \varepsilon x/\|x\|$. We claim, then, that $r^2(z) \le r^2 - \varepsilon^2$. To see this let $y \in S$, and consider the two cases $\langle y, x/\|x\|\rangle > \varepsilon$ and $\langle y, x/\|x\|\rangle \le \varepsilon$. In the first case a straightforward calculation shows that $\|y - z\|^2 \le r^2 - \varepsilon^2$. For the second case the cosine law shows that

$$\|y-z\|^2 = \|y-x\|^2 - \|x-z\|^2 + 2\langle y-z, x-z\rangle.$$

It can now be shown that $\langle y-z, x-z\rangle \le 0$, hence

$$\|y-z\|^2 \le r^2(x) - \|x-z\|^2.$$

Using the definition of $\varepsilon$ this then yields

$$\|y-z\|^2 \le r^2 - \varepsilon^2.$$

Thus $r^2(z) \le r^2 - \varepsilon^2$. However this contradicts the definition of $c$, and hence the lemma has been established. □

LEMMA 3.2. *For every $S \subseteq K$, $\sup\{r(T)\colon\ T \subset S$ and $T$ is finite$\} = r$.*

PROOF. Assume there is an $r' < r$ such that $r(T) < r'$ for every finite set $T \subset S$. The finite intersection property then shows that $\bigcap_{x\in S} \overline{B}(x, r') \ne \emptyset$, implying that $r = r(s) \le r'$ a contradiction. □

THEOREM 3.3 (KIRSZBRAUN). *Let $\{x_i\colon\ i \in I\}$ and $\{y_i\colon\ i \in I\}$ be sets in $\mathcal{H}$ and $\{r_i\colon\ i \in I,\ r_i > 0\}$ be a set of real numbers such that*

$$\|x_i - x_j\| \le \|y_i - y_j\| \qquad (i,j \in I).$$

*Then*

$$\bigcap_{i\in I} \overline{B}(x_i, r_i) = \emptyset \quad \textit{implies} \quad \bigcap_{i \in I} \overline{B}(y_i, r_i) = \emptyset.$$

PROOF. [**8**, p. 47].

LEMMA 3.4. *For each $S \subseteq K$, $r(S) \le r(f(S))$.*

PROOF. For a contradiction, assume that $\varepsilon = r(S) - r(f(S)) > 0$. By the definition of $r(S)$,

$$\bigcap_{s\in S} \overline{B}(s, r(S) - \varepsilon) = \emptyset.$$

As $f$ is noncontractive,

$$\|f(s_1) - f(s_2)\| \geq \|s_1 - s_2\| \quad \text{for all } s_1, s_2 \in S.$$

Kirszbraun's theorem implies that

$$\emptyset = \bigcap_{s \in S} \overline{B}(f(s), r(S) - \varepsilon) = \bigcap_{s \in S} \overline{B}(f(S), r(f(S))) = \{c(f(S))\}$$

leading to a contradiction. □

LEMMA 3.5. *Assume that $S \subseteq K$ satisfies $f(S) \subseteq S$. Then $r(S) = r(f(S))$ and $c(S) = c(f(S))$.*

PROOF. Because $f(S) \subseteq S$ we have $r(f(S)) \leq r(S)$. On the other hand Lemma 3.4 shows that $r(S) \leq r(f(S))$. Hence $r(S) = r(f(S))$. Now

$$f(S) \subseteq S \subseteq \overline{B}(c(S), r(S)) = \overline{B}(c(S), r(f(S))).$$

However $c(f(S))$ is the unique point in $\mathcal{H}$ such that $\overline{B}(c(f(S)),\ r(f(S))) \subseteq f(S)$. Thus $c(S) = c(f(S))$. □

LEMMA 3.6. *Let $S \subseteq K$ satisfy $f(S) \subseteq S$. Then for each $x \in K$, $\|f(x) - c(S)\| \geq \|x - c(S)\|$.*

PROOF. From Lemma 3.5, $c(S) = c(f(S))$ and $r(S) = r(f(S))$. Letting $c = c(S)$ and $r = r(S)$, assume for some $x \in K$ that $\|x - c\| > \|f(x) - c\|$. Let $\varepsilon = \|x - c\| - \|f(x) - c\|$. Now $\bigcap_{s \in S} \overline{B}(s, r) = \{c\}$, so

$$\phi = \bigcap_{s \in S} \overline{B}(s, r) \cap \overline{B}(x, \|x - c\| - \varepsilon).$$

By Kirszbraun's Theorem (Theorem 3.3)

$$\phi = \bigcap_{s \in S} \overline{B}(f(s), r) \cap \overline{B}(f(x), \|f(x) - c\|) = \{c\}.$$

Thus we have a contradiction, hence

$$\|x - c\| \leq \|f(x) - c\|. \quad \square$$

If $\mathcal{F} = \{f_\alpha\colon\ \alpha \in A\}$ is a commutative (or left reversible) semigroup then the set $A$ can be directed as follows. For $\alpha, \beta \in A$ define

$$\alpha \leq \beta \quad \text{if and only if } f_\alpha \mathcal{F} \supseteq f_\beta \mathcal{F}.$$

Thus for each $x$, $\mathcal{F}(x) = \{f_\alpha(x)\colon\ \alpha \in A\}$ is a net. Also, without loss of generality when finding fixed points, it may be assumed the identity is in $\mathcal{F}$.

COROLLARY 3.7. *If $\mathcal{F}\colon K \to K$ is a commutative semigroup of noncontractive mappings, $K$ a closed bounded convex set in $\mathcal{H}$, and $\mathcal{F}\colon\ S \subseteq K \to S$ then for $\alpha \geq \beta$,*

$$\|f_\alpha(x) - c(S)\| \geq \|f_\beta(x) - c(S)\|$$

*for each $x \in K$.*

PROOF. Because $f_\alpha \mathcal{F} \supseteq f_\beta \mathcal{F}$, and the identity is in $\mathcal{F}$, $f_\beta = f_\alpha f_\eta$, $\eta \in A$, and as $\mathcal{F}$ is commutative, $f_\beta(x) = f_\eta(f_\alpha(x))$. The corollary follows from Lemma 3.6. □

REMARK. This corollary is the only point in our argument where the commutativity of $\mathcal{F}$ is used. If $\mathcal{F}$ were right-reversible, rather than commutative, the corollary would be valid.

LEMMA 3.8. *For every $\varepsilon > 0$ there is a finite subset $T \subseteq S$ such that $\|c(S) - c(f(T))\| < \varepsilon$ for every noncontractive function $f$ with $f(S) \subseteq S$.*

PROOF. Let $\varepsilon > 0$ be given. Lemma 3.2 shows that there is a finite subset $T \subseteq S$ with $r^2(S) - r^2(T) < \varepsilon^2$. Let $f$ be an arbitrary noncontractive function with $f(S) \subseteq S$. From Lemma 3.1, we have

$$r^2(f(T)) + \|c(f(T)) - c(S)\|^2 \leq r^2(f(T), c(S)) \leq r^2(S).$$

Lemma 3.4 shows that $r(T) \leq r(f(T))$, so

$$r^2(T) + \|c(S) - c(f(T))\|^2 \leq r^2(S).$$

Hence $\|c(S) - c(f(T))\| < \varepsilon$, establishing the lemma. □

LEMMA 3.9. *Let $\varepsilon > 0$ be given. Then there is a finite subset $T \subseteq S$ such that for every linear functional $h$, $\|h\| = 1$, and each noncontractive function $f$ satisfying $f(S) \subseteq S$ it is true that for some $x \in T$,*

$$h(f(x)) - h(c(S)) < \varepsilon.$$

PROOF. By Lemma 3.8 there is a finite subset $T \subseteq S$ such that for every noncontractive function $f$, $\|c(S)-c(f(T))\| < \varepsilon$. As has been mentioned, $c(f(T)) \in \overline{\text{co}}(f(T))$, hence for some $x \in T$, $h(x) - h(c(f(T))) \leq 0$. Thus

$$h(x) - h(c(S)) \leq h(c(f(T))) - h(c(S)) < \varepsilon. \quad \square$$

By an arc $\gamma$ in $\mathcal{H}$ (or any metric space) we mean the image of a function $\gamma$: $[a, b] \to \mathcal{H}$ for some interval $[a, b]$ of $\mathbf{R}$. The length of an arc may be defined by purely metric means, without any differentiability assumption. We refer the reader to a book on metric geometry such as [**2**] for a discussion of arcs and their lengths. Let $l(\gamma)$ denote the arclength of (the image of) $\gamma$. If $f$ satisfies

$$k_1\|x - y\| \leq \|f(x) - f(y)\| \leq k_2\|x - y\|$$

and $\gamma$ lies in the domain of $f$, then $f \circ \gamma$ is an arc and

$$k_1 l(\gamma) \leq l(f(\gamma)) \leq k_2 l(\gamma).$$

LEMMA 3.10. *Let $\gamma(t)$, $a \leq t \leq b$ be an arc satisfying*

$$\frac{\langle \gamma(a), \gamma(b)\rangle}{\|\gamma(a)\|\,\|\gamma(b)\|} \leq \cos\theta, \qquad 0 \leq \theta \leq \pi,$$

*and $\|\gamma(t)\| \geq d$, $t \in [a, b]$. Then the length of $\gamma$ is at least $d \cdot \theta$.*

PROOF. Let $P$ be the projection of $\mathcal{H}\backslash\{0\}$ onto the sphere of radius $d$, defined by $P(x) = d \cdot x/\|x\|$. Certainly $P$ is nonexpansive on $\{x \in \mathcal{H}\colon \|x\| \geq d\}$ so $l(P(\gamma)) \leq l(\gamma)$. However $P(\gamma)$ is no shorter than the geodesic on $S$ joining $P(\gamma(a))$ to $P(\gamma(b))$, whose length is at least $d \cdot \theta$. □

THEOREM. *Let $\mathcal{F} = \{f_\alpha | \alpha \in A\}$ be a commutative semigroup (with identity) of self-mappings of a closed bounded convex set $K$ in a Hilbert space $\mathcal{H}$. Assume that for some $\delta > 0$,*

$$\|x - y\| \leq \|f_\alpha(x) - f_\alpha(y)\| \leq (\pi/2 - \delta)\|x - y\|.$$

*Then $\mathcal{F}$ has a fixed point.*

PROOF. Let $x_0 \in K$ be arbitrary, $x_\alpha = f_\alpha(x_0)$ and $S = \{x_\alpha | \alpha \in A\}$. Note that $f_\alpha(S) \subseteq S$ for all $\alpha \in A$. Let $c = c(S)$, $r = r(S)$. As mentioned earlier, $c \in \overline{\text{co}}\{x_\alpha\colon \alpha \in A\} \subseteq K$ (see [**5**]). Let $[c, x_\alpha]$ denote the arc $\{y\colon y = \lambda(c)+(1-\lambda)x_\alpha,\ 0 \leq \lambda \leq 1\}$. We now show that for some $x \in \bigcup_{\alpha\in A}[c, x_\alpha]$, $\|c - f_\alpha(x)\| \leq (1 - \delta/\pi)r$, for all $\alpha \in A$.

Clearly if $r = 0$, then $x$ may be taken to be $c$ (which is a fixed point of $\mathcal{F}$). Thus, for a contradiction, assume that $r > 0$ and that $\sup\{\|c - f_\alpha(x)\|\colon \alpha \in A\} > (1 - \delta/\pi)r$ for all $x \in \bigcup_{\alpha\in A}[c, x_\alpha]$.

Let $\varepsilon > 0$ be given. By Lemma 3.9 choose a finite set $T \subseteq S$ such that for any $\alpha \in A$, and any linear functional $h$, $\|h\| = 1$, there is a $z \in T$ with $h(f_\alpha(z)) - h(c) < \varepsilon$. Since $\bigcup_{y\in T}[c, y]$ is compact, Corollary 3.7 implies that for some $\eta \in A$ and for every $x \in \bigcup_{y\in T}[c, y]$,

$$\|c - f_\eta(x)\| > \left(1 - \frac{\delta}{\pi}\right) r.$$

Define

$$h(x) = \left\langle \frac{f_\eta(c) - c}{\|f_\eta(c) - c\|}, x \right\rangle$$

and let $z \in T$ satisfy $h(f_\eta(z)) - h(c) < \varepsilon$. Thus

$$\left\langle \frac{f_\eta(c) - c}{\|f_\eta(c) - c\|}, \frac{f_\eta(z) - c}{\|f_\eta(z) - c\|} \right\rangle < \frac{\varepsilon}{(1 - \delta/\pi)r}.$$

Hence angle $f_\eta(c)cf_\eta(z)$ is greater than $\cos^{-1}(\varepsilon/(1 - \delta/\pi)r)$ and by Lemma 3.10

$$l(f_\eta[c, z]) > \left(1 - \frac{\delta}{\pi}\right) r \cos^{-1}\left(\frac{\varepsilon}{(1 - \delta/\pi)r}\right).$$

As $\varepsilon > 0$ was arbitrary, it must be that

$$\sup\{l(f_\eta[c, z])\colon \eta \in A, z \in S\} \geq \left(\frac{\pi}{2} - \frac{\delta}{2}\right) r. \tag{3.3}$$

However, as $\|f_\eta(x) - f_\eta(y)\| \leq (\frac{\pi}{2} - \delta)\|x - y\|$ for all $x, y \in K$ and $\eta \in A$, then for each $\eta \in A$ and $z \in S$,

$$l(f_\eta[c, z]) \leq \left(\frac{\pi}{2} - \delta\right) \|c - z\| \leq \left(\frac{\pi}{2} - \delta\right) r. \tag{3.4}$$

Combining 3.3 and 3.4, we reach the desired contradiction, and conclude that for some $x \in \bigcup_{\alpha\in A}[c, x_\alpha]$,

$$\|c - f_\alpha(x)\| \leq \left(1 - \frac{\delta}{\pi}\right) r \quad \text{for all } \alpha \in A.$$

Let $y_0 \in K$ be arbitrary and let $r = r(\{f_\alpha(y_0)\colon \alpha \in A\})$. Assume that for $n < k$, $y_n$ has been defined so that

$$\|y_n - y_{n-1}\| \leq 2\left(1 - \frac{\delta}{\pi}\right)^{n-1} r \tag{3.5}$$

and

$$r\{f_\alpha(y_n)\colon \alpha \in A\} \leq \left(1 - \frac{\delta}{\pi}\right)^n r. \tag{3.6}$$

Using the above argument with $x_0 = y_{k-1}$ we find a point $y_k = x$ such that

$$\{f_\alpha(y_k)\colon\ \alpha \in A\} \subseteq \overline{B}\left(c(\{f_\alpha(y_{k-1})\colon\ \alpha \in A\}), \left(1 - \frac{\delta}{\pi}\right)^k r\right), \tag{3.7}$$

and hence (3.6) is satisfied with $n = k$, and

$$\|y_k - c\{f_\alpha(y_{k-1})\colon\ \alpha \in A\}\| \le \left(1 - \frac{\delta}{\pi}\right)^k r \le \left(1 - \frac{\delta}{\pi}\right)^{k-1} r \tag{3.8}$$

(recall we assume that $\mathcal{F}$ contains the identity). Also,

$$\|y_{k-1} - c\{f_\alpha(y_{k-1})\colon\ \alpha \in A\}\| \le r(\{f_\alpha(y_{k-1})\colon\ \alpha \in A\}) \le \left(1 - \frac{\delta}{\pi}\right)^{n-1} r. \tag{3.9}$$

Hence 3.8 and 3.9 together show 3.5 is satisfied with $n = k$.

We conclude, using the continuity of the $f_\alpha$, that $\{y_n\}$ is a Cauchy sequence, and its limit is a fixed point of $\mathcal{F}$. □

REMARK. The above theorem holds for a right reversible, rather than commutative, semigroup. See the remark following Corollary 3.6.

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DEPARTMENT OF MATHEMATICS, DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, CANADA

*Current address*: Department of Mathematics, Mount Allison University, Sackville, New Brunswick, Canada