REGULARITY OF BANACH ALGEBRAS GENERATED BY ANALYTIC SEMIGROUPS SATISFYING SOME GROWTH CONDITIONS

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ABSTRACT. We show that if a commutative complex Banach algebra A is generated by a nonzero analytic semigroup $(a^t)_{\operatorname{Re} t > 0}$ satisfying

$$\int_{-\infty}^{+\infty} \frac{\log^+ \|a^{1+it}\|}{1+t^2} dt < +\infty,$$

then A is regular in Shilov's sense.

1. Introduction. Recall that a commutative (nonradical) complex Banach algebra A is said to be regular in Shilov's sense, or regular for short, if, given $\alpha_0 \in \hat{A}$ and a neighbourhood $\mathcal{U}$ of α_0 in $\hat{A}$ there exists $b \in A$ with $\alpha_0(b) \neq 0$ and $\alpha(b) = 0$ for every $\alpha \notin \mathcal{U}$. (Here we denote by $\hat{A}$ the carrier space of A equipped with the Gelfand topology.)

The classical proofs of the Wiener tauberian theorem for the convolution algebra $L^1(R)$ use the fact that $L^1(R)$ is regular. The Wiener tauberian theorem extends to the weighted algebras $L^1(R, e^{\phi(t)})$ where ϕ is a nonnegative, measurable function on R satisfying $\phi(s+t) \leq \phi(s) + \phi(t)$ ($s, t \in R$) and

$$\int_{-\infty}^{+\infty} \frac{\phi(t)}{1+t^2} dt < +\infty$$

(the first condition just ensures that $L^1(R, e^{\phi(t)})$ is stable under convolutions), and Beurling's original proof of this result [1] is also based upon the regularity of $L^1(R, e^{\phi(t)})$. There are some noncommutative extensions of Wiener's tauberian theorem to the group algebras $L^1(G)$ where G is, for example, a nilpotent Lie group. In that case it is true that if I is a two-sided ideal of $L^1(G)$ such that $I \neq L^1(G)$ then the quotient algebra $L^1(G)/I$ is not radical (in other terms $L^1(G)$ possesses the "weak Wiener property" in the sense of Leptin [7]). This fact is proved by Hulanicki in [5] using the regularity of the commutative subalgebra of $L^1(G)$ generated by the "heat semigroup" (see [7] for a discussion of noncommutative one-sided or two-sided extensions of the tauberian theorem).

In [4] the first author gives a proof of the Wiener tauberian theorem for $L^1(R)$ based upon an application of the Ahlfors-Heins theorem [2, Theorem 7-2-6] to the heat semigroup. Dales and Hayman give in [3] a proof of the tauberian theorem for some Beurling algebras which also involves the growth on vertical lines of the heat

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semigroup and some more sophisticated results in the theory of analytic functions. Hulanicki [6] observed that the argument of [4] applied to the heat semigroup on a nilpotent Lie group G gives also a proof of the weak Wiener property for $L^1(G)$. In fact the method used in [4] can be applied to Banach algebras A which possess an analytic semigroup $(a^t)_{\operatorname{Re} t > 0}$ such that $[a^t A]^- = A$ ($\operatorname{Re} t > 0$),

$$\int_{-\infty}^{+\infty} \frac{\log^+ \|a^{1+iy}\|}{1+y^2} dy < +\infty.$$

As shown in [8, Lemma 5-7] such an analytic semigroup is of exponential type in the half-plane $\operatorname{Re} z \geq 3$, and the Ahlfors-Heins theorem shows then that every proper two-sided ideal of A is contained in a primitive ideal of A [8, Theorem 5-6]. This use of the Ahlfors-Heins theorem avoids any reference to regularity.

We show here in the other direction that the above growth condition implies that a large commutative subalgebra of A is regular. More precisely we have the following theorem.

THEOREM 1. *Let $(a^t)_{\operatorname{Re} t > 0}$ be a nonzero analytic semigroup in a Banach algebra A . If*

$$\int_{-\infty}^{\infty} \frac{\log^+ \|a^{1+it}\|}{1+t^2} dt < +\infty,$$

then the closed commutative subalgebra B of A generated by $\{a^t\}_{\operatorname{Re} t > 0}$ is regular.

2. Proof of the theorem. Note that Theorem 5-6 of [8] shows that B is not radical. We will need two lemmas.

LEMMA 1. *Put $\omega(t) = \sup(1, \|a^{2+it}\|)$ ($t \in \mathbb{R}$). Then for every $y_0 \in \mathbb{R}$ and every $\eta > 0$ there exists an element f of the Banach space $L^1(\mathbb{R}, \omega)$ whose Fourier transform $\hat{f}$ satisfies $\hat{f}(y_0) = 1$, $\hat{f}(y) = 0$ if $|y - y_0| \geq \eta$.*

PROOF. Put $\omega_0(t) = \sup(1, \|a^{1+it}\|)$ ($t \in \mathbb{R}$). Note that $L^1(\mathbb{R}, \omega_0)$ and $L^1(\mathbb{R}, \omega)$ are contained in $L^1(\mathbb{R})$, so that the convolution product $g_1 * g_2$ is well defined for every $g_1, g_2 \in L^1(\mathbb{R}, \omega_0)$ and the Fourier transform $\hat{f}$ of f is well defined for every $f \in L^1(\mathbb{R}, \omega) \cup L^1(\mathbb{R}, \omega_0)$. As $\omega(s+t) \leq \omega_0(s)\omega_0(t)$ ($s, t \in \mathbb{R}$), it follows from a standard application of Fubini's theorem that $g_1 * g_2 \in L^1(\mathbb{R}, \omega)$ for every $g_1, g_2 \in L^1(\mathbb{R}, \omega_0)$.

We have $\omega_0(t) = e^{\log^+ \|a^{1+it}\|}$ ($t \in \mathbb{R}$) where

$$\int_{-\infty}^{+\infty} \frac{\log^+ \|a^{1+it}\|^2}{1+t^2} dt < +\infty.$$

A classical argument recalled in [3] shows then that there exists $g_1 \in L^1(\mathbb{R}, \omega_0)$ such that $\hat{g}_1(y_0) = 1$, $\hat{g}_1(y) = 0$ for every $y \in (-\infty, y_0 - \eta]$ and $g_2 \in L^1(\mathbb{R}, \omega_0)$ such that $\hat{g}_2(y_0) = 1$, $\hat{g}_2(y) = 0$ for every $y \in [y_0 + \eta, \infty)$. Then $f = g_1 * g_2$ belongs to $L^1(\mathbb{R}, \omega)$ and satisfies the conditions of the lemma.

LEMMA 2. *For every character α of B there exists a unique real number $\tilde{\alpha}$ such that $\alpha(a^t) = \exp(i\tilde{\alpha}t)$ ($\operatorname{Re} t > 0$).*

PROOF. The map $\theta: t \mapsto \alpha(a^t)$ is analytic, and $\theta(t+t') = \theta(t)\theta(t')$ if $\operatorname{Re} t > 0$, $\operatorname{Re} t' > 0$. So there exists $\tilde{\alpha} \in \mathbb{C}$ such that $\alpha(a^t) = \exp(i\tilde{\alpha}t)$ ($\operatorname{Re} t > 0$). Put

$\tilde{\alpha} = \lambda + i\mu$ where λ and μ are real. Then $|\alpha(a^{1+it})| = \exp(\lambda - t\mu)$ ($t \in \mathbb{R}$) and as

$$\int_{-\infty}^{+\infty} \frac{\log^+ |\alpha(a^{1+it})|}{1+t^2} dt < +\infty$$

we must have $\mu = 0$. So $\tilde{\alpha}$ is a real number. If another real number δ satisfies the condition $\alpha(a^t) = \exp(i\delta t)$ ($\operatorname{Re} t > 0$) then $(\tilde{\alpha} - \delta)/2\pi t \in \mathbb{Z}$ for every t , so that $\tilde{\alpha} = \delta$ and α is unique. This proves the lemma.

We can now prove the theorem. Let $\alpha_0 \in \hat{B}$ and let $\mathcal{U}$ be a neighborhood of α_0 in $\hat{B}$. Note that if a sequence (β_n) of elements of $\hat{B}$ satisfies

$$|\tilde{\beta}_n - \tilde{\alpha}_0| \xrightarrow{n \rightarrow \infty} 0$$

then $\alpha_0(a^t) = \lim_{n \rightarrow \infty} \beta_n(a^t)$ for every t , so that (β_n) converges towards α_n in $\hat{B}$. We thus see that there exists $\eta > 0$ such that $|\tilde{\alpha}_0 - \tilde{\beta}| \geq \eta$ for every $\beta \notin \mathcal{U}$.

It follows from Lemma 1 that there exists $f \in L^1(\mathbb{R}, \omega)$ such that

$$\int_{-\infty}^{+\infty} f(t)e^{it\tilde{\alpha}_0} dt = 1 \quad \text{and} \quad \int_{-\infty}^{+\infty} f(t)e^{it\lambda} dt = 0$$

if $|\lambda - \tilde{\alpha}_0| \geq \eta$. Then

$$\int_{-\infty}^{+\infty} |f(t)| \|a^{2+it}\| dt < +\infty,$$

and the Bochner integral

$$\int_{-\infty}^{+\infty} f(t)a^{2+it} dt$$

defines an element b of B .

As Bochner integrals commute with continuous linear forms, we have

$$\begin{aligned} \alpha_0(b) &= \int_{-\infty}^{+\infty} f(t)\alpha_0(a^{2+it}) dt \\ &= e^{2\tilde{\alpha}_0} \int_{-\infty}^{+\infty} f(t)e^{it\tilde{\alpha}_0} dt = e^{2\tilde{\alpha}_0} \neq 0. \end{aligned}$$

We see similarly that $\beta(b) = 0$ for every $\beta \notin \mathcal{U}$, as $|\tilde{\beta} - \tilde{\alpha}_0| \geq \eta$. This concludes the proof of the theorem.

REMARK. If a semigroup $(a^t)_{\operatorname{Re} t > 0}$ in a Banach algebra A satisfies the growth condition of Theorem 1 then $a^t \in \operatorname{Sp}(a^n)_{n \in \mathbb{N}}$ for every t , and the closed subalgebra of A generated by the semigroup equals the closed subalgebra generated by a^1 . To see this note that

$$\int_{-\infty}^{+\infty} \frac{\log^+ |l(a^{1+it})|}{1+t^2} dt < +\infty$$

for every continuous linear form l on A . As the function $t \mapsto l(a^t)$ is analytic for $\operatorname{Re} t \geq 1$ we must have $l(a^t) = 0$ for every t if $l(a^n) = 0$ for every $n \geq 1$ [2, Theorem 9-3-8].

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