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COUNTEREXAMPLES TO SEVERAL PROBLEMS CONCERNING G_δ-EMBEDDINGS

N. GHOUSSOUB AND B. MAUREY[1]

ABSTRACT. Let JT (resp. JH) be the James-tree space (resp. the Hagler space). We observe that two canonical subspaces of JT* and JH*, G_δ-embed in l_2 even though they fail the Radon-Nikodym property. Such G_δ-embeddings cannot be the product of a finite number of semi-embeddings. This answers negatively several questions of Bourgain-Rosenthal [**1**] and Rosenthal [**8**].

Introduction. Let X and Y be two separable Banach spaces and let $S: X \to Y$ be a given operator. S is called a *semi-embedding* if S is one-to-one and $S(\text{Ball}(X))$ is closed. S is called a *G_δ-embedding* if S is one-to-one and $S(F)$ is a G_δ for any closed bounded subset F of X. Note that if Z is another Banach space and S_1, S_2 are two operators such that $S_1: X \to Y$ is a semi-embedding and $S_2: Z \to X$ is an isomorphic embedding then S_1S_2 is a G_δ-embedding. In [**1** and **3**] stability properties of the following type were investigated: if $S: X \to Y$ is a G_δ-embedding and T is an operator from L_1 into X, what are then the properties (P) such that:

T satisfies (P) if and only if ST satisfies (P)?

In [**3**] it is shown that the answer is positive for (P) = Dunford-Pettis and (P) = norm-sign preserving. In [**1**] it is shown that the answer is positive if (P) = representable provided S is a semi-embedding or, more generally, a composition of a finite number of semi-embeddings. In this note we construct G_δ-embeddings from two spaces failing the Radon-Nikodym property into l_2. Such a G_δ-embedding cannot be the product of semi-embeddings. Hence this gives a negative answer to questions (1) and (2) of Bourgain-Rosenthal [**1**]. This also solves negatively the local problem formulated by Rosenthal in [**8**]: namely, the existence of a relatively compact measure convex, L_1-convex G_δ-subset of l_2 without the R.N.P. as defined in [**8**] for nonclosed sets.

In the following proposition we give a general procedure to construct G_δ-embeddings which are not the product of a finite number of semi-embeddings.

PROPOSITION (1). *Let Y be a separable Banach space and let X be a separable subspace of Y^* such that*

(i) $\text{Ball}(X)$ *is a weak*-G_δ in* $\text{Ball}(Y^*)$,

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(ii) Ball(X) *contains a closed convex set K with no extreme points.*
Then for every dense range compact operator T from l_2 into Y, the restriction S of T^* to X is a G_δ-embedding of X into l_2 which is not the product of a finite number of semi-embeddings. Moreover $S(K)$ is a measure convex, L_1-convex, relatively norm-compact G_δ-subset of l_2 without any extreme point (hence without the R.N.P.).

PROOF. Suppose $B_{Y^*} \setminus B_X = \bigcup_n K_n$ where the K_n's are weak*-compact subsets of Y^*. We shall prove that for any closed subset F in B_X, the set $B_{Y^*} \setminus F$ is also a weak*-K_σ. Indeed, for each x in X, denote by $r(x)$ the distance $d(x, F)$ of x to F and let $D(x) = \{y \in X; \|y - x\| < r(x)/2; \|y\| \leqslant 1\}$. These are open sets in the polish space Ball(X), hence, since $B_X \setminus F = \bigcup_{x \in B_X \setminus F} D(x)$, there exists a countable subfamily (x_n) such that $B_X \setminus F = \bigcup_n D(x_n)$.

Let now $C(x) = \{y \in Y^*; \|y - x\| \leqslant r(x)/2; \|y\| \leqslant 1\}$. Note that each $C(x)$ is weak*-compact in Y^*. We claim that $F \cap (\bigcup_n C(x_n)) = \varnothing$: if not, there will be y in F and an x_n in $B_X \setminus F$ such that $\|y - x_n\| \leqslant \frac{1}{2}r(x_n) = \frac{1}{2}d(x_n, F)$, which is absurd. Finally we get

$$B_{y^*} \setminus F = \Big(\bigcup_n K_n\Big) \cup \Big(\bigcup_n C(x_n)\Big),$$

which is a weak*-K_σ.

Let now T be a dense range bounded linear operator from l_2 into Y. It follows that T^*: $Y^* \to l_2$ is one-to-one. Moreover, the image of every weak*-compact subset in Y^* is weakly compact in l_2, from which follows that the restriction S of T^* to X is a G_δ-embedding.

Since X fails the Radon-Nikodym property, the above discussion gives that S cannot be the product of a finite number of semi-embeddings. Moreover, $S(K)$ has no extreme points since S is one-to-one, hence it fails the R.N.P. as defined in [**8**] for nonclosed sets. It also has the convexity properties mentioned above since it is the continuous image of a closed convex set. See [**8**] for more details.

We shall now give two examples of pairs of spaces (X, Y^*) verifying the hypothesis of the proposition. These spaces will clearly provide the required counterexamples. The first one is the couple (B, JT^*), where JT is the James-tree space [**6**] and B is its predual, studied by Lindenstrauss-Stegall [**7**]. It was shown by Edgar-Wheeler [**2**] that B is a weak*-G_δ in its bidual JT* and it is well known that B does not have the Radon-Nikodym property. Actually, it can be shown that B does not have the Krein-Milman property. It follows then that the pair (B, JT^*) verifies the hypothesis of the proposition. For more details we refer the reader to [**2** and **7**].

We recall the construction of the second example, which is the pair (F, JH^*) where F is the subspace of the dual of Hagler's space JH [**5**], also studied by Bourgain-Rosenthal [**1**], since it is easier to describe and since it turns out to be the "general case": indeed, in a forthcoming paper [**4**] we show that every Banach space that G_δ-embeds in l_2 by means of a compact operator sits in the dual of a separable Banach space the way F sits in JH*.

The space JH consists of all functions x from the infinite tree $T = \{(n, i); n = 0, 1, 2; 0 \leqslant i < 2^n\}$ into the reals so that $\|x\| = \sup \sum_{i=1}^{r} |S_i^*(x)| < \infty$, where the sup is taken over all families $S_1, \ldots, S_r$ of admissible segments in T. (The segments $S_1, \ldots, S_r$ are called admissible provided they have the same starting-level, the same ending-level and are mutually disjoint.) If S is a segment, then

$$S^*(x) = \sum_{(n,i) \in S} x(n, i).$$

Let $e_{n,i}$ be the unit vectors of JH (i.e. the elements defined by $e_{n,i}(m, j) = \delta_{n,m}\delta_{i,j}$). Let F be the closed linear span in JH* of the biorthogonal functionals $(e^*_{n,i})$ to $(e_{n,i})$ (i.e. $e^*_{n,i}(e_{m,j}) = \delta_{n,m}\delta_{i,j}$). Denote by Γ the uncountable set of branches of T. J. Hagler proved [5] the following facts:

(a) For every branch γ, and $x^* \in \mathrm{JH}^*$, $\lim_{(n,i)\in\gamma} x^*(e_{n,i})$ exists.

(b) The space F is the kernel of the map S: $\mathrm{JH}^* \to c_0(\Gamma)$ defined by $Sx^*(\gamma) = \lim_{(n,i)\in\gamma} x^*(e_{n,i})$.

It then follows easily that

$$F = \left\{ x^* \in \mathrm{JH}^*;\ \liminf_{n\to\infty}\left(\max_{0 \leqslant i < 2^n} |x^*(e_{n,i})| \right) = 0 \right\}.$$

Hence $\mathrm{Ball}(\mathrm{JH}^*) \setminus \mathrm{Ball}(F) = \bigcup_m K_m$, where for each m,

$$K_m = \bigcap_{n \geqslant m} \left\{ x^* \in \mathrm{JH}^*;\ \|x^*\| \leqslant 1;\ \max_{0 \leqslant i < 2^n} |x^*(e_{n,i})| \geqslant \frac{1}{m} \right\}$$

is weak*-compact.

Following Bourgain-Rosenthal [1] we define for each $(n, i) \in T$ and each integer k, the vector

$$f^k_{n,i} = 2^{-k} \sum_S \sum_{(l,m)\in S} e^*_{(l,m)},$$

where the first sum runs over all segments S with starting level 1, ending level $n + k$ and passing through (n, i). The sequence $(f^k_{n,i})_k$ is norm convergent since $\|f^{k+1}_{n,i} - f^k_{n,i}\| = 2^{-k-1}$. Let now $f_{n,i} = \lim_{k\to\infty} f^k_{n,i}$. It is easy to see that $\{f_{n,i};\ n \in \mathbf{N},\ 0 \leqslant i < 2^n\}$ is a 1-tree: that is, for each $(n, i) \in T$ we have

$$f_{n,i} = \frac{1}{2}(f_{n+1,2i} + f_{n+1,2i+1}) \quad \text{and} \quad \|f_{n+1,2i} - f_{n+1,2i+1}\| \geqslant 1.$$

Let now D be the weak*-closed convex hull of $L = \{ f_{n,i}:\ (n, i) \in T\}$ and let $K = D \cap F$: that is,

$$K = \left\{ x^* \in D,\ \lim_{(n,i)\in\gamma} x^*(e_{n,i}) = 0 \quad \text{for all } \gamma \in \Gamma \right\}.$$

Note that K is a closed convex subset of D such that every extreme point of K must be an extreme point of D, since for every $x^* \in D$ we have $\lim_{(n,i)\in\gamma} x^*(e_{n,i}) \geqslant 0$ for each $\gamma \in \Gamma$. On the other hand, Krein's theorem guarantees that the extreme points of D are in the weak*-closure $\overline{L}$ of L. Since no element in L can be extreme, we have that $\mathrm{Ext}(K) \subseteq \mathrm{Ext}(D) \subseteq \overline{L}\setminus L$. On the other hand, it is easy to see that for each $x^* \in \overline{L}\setminus L$, we have $\lim_{n\to\infty} \sup_{1 \leqslant i < 2^n} |x^*(e_{n,i})| = 1$, hence $K \cap (\overline{L}\setminus L) = \varnothing$ and K has no extreme points. It follows from the proposition that for every compact dense range operator T: $l_2 \to \mathrm{JH}$, the restriction $S = T^*_F$: $F \to l_2$ is a G_δ-embedding and $S(K)$ is a relatively compact, measure convex, L_1-convex G_δ-subset of l_2 with no extreme points.

ADDED IN PROOF. In a forthcoming paper [**4**] we give a complete characterization of the spaces that G_δ-embed in l_2 by means of compact operators. In particular, we show that spaces with the Radon-Nikodym property do G_δ-embed in l_2. It follows that the $\mathscr{L}_\infty$-spaces constructed by Bourgain-Delbaen are also examples of spaces G_δ-embeddable into l_2. In view of the results of Bourgain-Rosenthal [**1**], such G_δ-embeddings cannot be the product of a finite number of semi-embeddings.

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER V6T 1W5, BRITISH COLUMBIA, CANADA

DÉPARTEMENT DE MATHÉMATIQUE, UNIVERSITÉ DE PARIS VII, PARIS, FRANCE