SEQUENTIAL COREFLECTIONS OF STRATIFIABLE SPACES

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ABSTRACT. We prove that the sequential coreflection of a stratifiable space which is the union of countably many closed k -spaces is stratifiable.

Several years ago E. Michael [10] showed that among spaces which are the union of countably many closed k -spaces, the presence of a σ -locally finite k -network allows the preservation of certain topological properties (normality, paracompactness, stratifiability) when forming $k(X)$. In this brief note we show that with use of the Lemma below essentially the same proof will work in k -semistratifiable spaces.

DEFINITIONS. Let $(X, \mathcal{T})$ be a Hausdorff topological space. A function $S: N \times \mathcal{T} \rightarrow \{A \in \mathcal{P}(X): A \text{ is closed in } X\}$ is a *semistratification* [3] of X if the following hold.

- (a) If $U \in \mathcal{T}$, then $U = \bigcup_{n \in \mathbb{N}} S(n, U)$.
- (b) If $U \in \mathcal{T}$ and $n \in \mathbb{N}$, then $S(n, U) \subset S(n+1, U)$.
- (c) If $U, V \in \mathcal{T}$ with $U \subset V$, and if $n \in \mathbb{N}$, then $S(n, U) \subset S(n, V)$.

A semistratification S of X is said to be a *k -semistratification* [9] of X if:

- (d) If $U \in \mathcal{T}$ and K is compact with $K \subset U$, then there is an $n \in \mathbb{N}$ so that $K \subset S(n, U)$; or equivalently:
- (d') If $U \in \mathcal{T}$ and Z is a sequence converging to a member of U , then there is an $n \in \mathbb{N}$ so that $S(n, U)$ contains a tail of Z .

A collection $\mathcal{F}$ of ordered pairs $F = (F_1, F_2)$ of subsets of X with $F_1 \subset F_2$ for all $F \in \mathcal{F}$ is called a *pair-cs-network* for X if for every open set U of X and a sequence Z converging to a point of U , there is an $F \in \mathcal{F}$ so that $F_2 \subset U$ and F_1 contains a tail of Z . One can show that a space X is k -semistratifiable if and only if X has a σ -cushioned pair-cs-network (cf. [2, 6, 8]). Hence any space with a σ -locally finite k -network is k -semistratifiable.

Note that because points are G_δ in semistratifiable spaces the notions of sequential space and k -space (as well as sequential coreflection and k -coreflection) coincide for semistratifiable spaces.

A subset W of a topological space X is said to be a *sequential neighborhood* of $x \in X$ if every sequence converging to x is eventually in W .

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LEMMA. Every k -seminstratifiable space X has a σ -cushioned pair-cs-network that respects sequential neighborhoods; i.e. if W is a sequential neighborhood of x and Z is a sequence converging to x , then there is an $F \in \mathcal{F}$ so that $F_2 \subset W$ and F_1 contains a tail of Z .

PROOF. Let $\mathcal{G} = \bigcup_{n \in \mathbb{N}} \mathcal{G}_n$ be a pair-cs-network for X , where the families $\mathcal{G}_n$ are cushioned. For each $n \in \mathbb{N}$ and $y \in X$, let $U(n, y)$ be an open neighborhood of y so that $U(n, y) \cap \bigcup \{G_1 : (G_1, G_2) \in \mathcal{G}_n, y \in X \setminus G_2\} = \emptyset$ and $U(n+1, x) \subset U(n, x)$. For each $W \subset X$ and $n \in \mathbb{N}$, let

$$F(n, W) = X \setminus \bigcup \{U(n, y) : y \in X \setminus W\}.$$

One may readily verify that for each n the family $\mathcal{F}_n = \{(F(n, W), W) : W \subset X\}$ is a cushioned family.

To show that $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ is a pair-cs-network that respects sequential neighborhoods, suppose not. Then there is an $x \in X$, a sequential neighborhood W of x , and a sequence $Z = \langle x_k : k \in \mathbb{N} \rangle$ converging to x so that no $F(n, W)$ contains a tail of Z . This allows us to pick a subsequence $\langle x_{k(n)} : n \in \mathbb{N} \rangle$ so that for all n , $x_{k(n)} \notin F(n, W)$. For every $n \in \mathbb{N}$, there is a $y_{k(n)} \in X \setminus W$ so that $x_{k(n)} \in U(n, y_{k(n)})$. We claim that $\langle y_{k(n)} : n \in \mathbb{N} \rangle$ converges to x , violating our assumption that W is a sequential neighborhood of x .

Indeed, if $\langle y_{k(n)} : n \in \mathbb{N} \rangle$ did not converge to x , then we could find a neighborhood V of x and a subsequence $\langle y_{k(n(m))} : m \in \mathbb{N} \rangle$ so that $V \cap \{y_{k(n(m))} : m \in \mathbb{N}\} = \emptyset$. Find $r \in \mathbb{N}$ and $G \in \mathcal{G}_r$ so that $G_2 \subset V$ and G_1 contains a tail of Z . Then find $m \in \mathbb{N}$ so that $n(m) \geq r$ and $x_{k(n(m))} \in G_1$. Because $y_{k(n(m))} \notin G_2$ we have that $U(r, y_{k(n(m))}) \cap G_1 = \emptyset$; thus $U(n(m), y_{k(n(m))}) \cap G_1 = \emptyset$, contradicting that $x_{k(n(m))} \in U(n(m), y_{k(n(m))})$.

The proof of the following Theorem is essentially that suggested in [10].

THEOREM. If the stratifiable space X is the union of countably many closed k -spaces, then $k(X)$ is stratifiable.

PROOF. It follows from the Lemma that $k(X)$ has a k -seminstratification S in which the sets $S(n, U)$ are closed in X . Let D be a monotone normality operator [7] for the stratifiable space X . We will show that $k(X)$ is also monotonically normal and consequently stratifiable [7, Theorem 2.5]. Write $X = \bigcup_{n \in \mathbb{N}} X_n$, where each X_n is a closed (in X) k -space. Let A and B be disjoint closed sets in $k(X)$. For every n , $A_n = A \cap X_n$ and $B_n = B \cap X_n$ are closed in X_n and so closed in X . Let

$$U_n = \bigcap_{m \in \mathbb{N}} D(A_n \cup S(m, X \setminus B), B_m) \setminus B$$

and

$$V_n = \bigcap_{m \in \mathbb{N}} D(B_n \cup S(m, X \setminus A), A_m) \setminus A.$$

By [10, Lemma 5.3] each U_n is open in the subspace $X \setminus B$ of $k(X)$, hence open in $k(X)$. Plainly the X -closure of U_n , and hence the $k(X)$ -closure of U_n , misses B .

Analogous things can be said of the V_n 's, so that

$$U = \bigcup_{n \in \mathbb{N}} \left[U_n \setminus \bigcup_{m \leq n} \text{cl}_{k(X)} V_m \right] \quad \text{and} \quad V = \bigcup_{n \in \mathbb{N}} \left[V_n \setminus \bigcup_{m \leq n} \text{cl}_{k(X)} U_m \right]$$

are disjoint $k(X)$ -open sets containing A and B , respectively. Tracing our argument with $A \subset A'$ and $B \supset B'$ would yield $U \subset U'$ and $V \supset V'$ and so $k(X)$ is monotonically normal, as desired.

The hypothesis that X be the union of countably many closed k -spaces cannot be omitted in the previous Theorem. The space Z of [4] is (as is easily seen) an M_1 -space. So while Z^2 is M_1 , $k(Z^2)$ is not regular [4, Theorem C].¹ We do not know if "stratifiable" may be replaced by " M_1 " in our theorem, although it is known that a stratifiable space which is the union of countably many closed metrizable spaces is M_1 [5].

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¹Stratifiability was claimed in [1] for an example given in [10] of a σ -compact cosmic space X so that $k(X)$ is not regular. By our Theorem this claim is incorrect.