A REMARK ON THE MAXIMAL DILATATION
OF A QUASICONFORMAL MAPPING

MATTI VUORINEN

ABSTRACT. A lower bound for the maximal dilatation of a quasiconformal self-mapping of a uniform domain G in R^n with connected boundary is proved provided that the boundary is kept pointwise fixed by the mapping and that a prescribed point a is mapped on another point b in G. The lower bound is given in terms of the quasihyperbolic distance from a to b.

1. Introduction. Let G be a proper subdomain of R^n $(n \geq 2)$, let $a, b \in G$ be given, and let $f \colon \overline{G} \to \overline{G}$ be a quasiconformal (qc) homeomorphism such that $f(a) = b$ and $f(x) = x$ for $x \in \partial G$. In this note we shall give a lower bound for the maximal dilatation $K(f)$ of f.

Gehring and Palka [3] have proved that there exists a qc mapping $g \colon \overline{G} \to \overline{G}$ with the above properties such that

$$(1.1) \qquad \log K(g) \leq 2(n-1) k_G(a, b),$$

where $k_G(a, b)$ is the quasihyperbolic distance ([3]) from a to b. Here the following lower bound will be established

$$(1.2) \qquad K(f) \geq c(n, G) k_G(a, b)^n$$

where $c(n, G)$ is a positive number depending only on n and G. The estimate (1.2) is valid under the following two assumptions (i) G is a uniform domain with connected boundary; (ii) $k_G(a, b)$ exceeds a bound, depending on the parameters in (i). Uniform domains were introduced by Martio and Sarvas [4] and have been studied by Gehring and Osgood [2].

For the dimension $n = 2$ and G = the unit disc, the exact bound in (1.2) has been determined by Teichmüller [5]. An n-dimensional counterpart of Teichmüller's result under an additional symmetry assumption was obtained by Anderson and Vamanamurthy [1]. The proof of (1.2) is based on a characterization of uniform domains due to Gehring and Osgood [2]. I am grateful to Professor F. W. Gehring for bringing this problem to my attention, to Professor G. D. Anderson and Professor J. Sarvas for conversations, and to the referee for some remarks.

2. Quasihyperbolic metric. Let $G \subset R^n$ be a domain, $G \neq R^n$, and let $d(x) = \operatorname{dist}(x, \partial G) \in (0, \infty)$ for $x \in G$. The *quasihyperbolic distance* between a and b in G is defined by (cf. [3])

$$k_G(a, b) = \inf_{\gamma} \int_{\gamma} d(x)^{-1} \, ds,$$

where γ runs through all rectifiable curves in G with $a, b \in \gamma$. It was proved by Gehring and Palka [3] that k_G is a metric in G, which defines the Euclidean topology of G. Several other properties of k_G can be found in [3 and 4].

A domain $G \subset R^n$ is said to be uniform (cf. [2, 4]) if there exist constants $\tilde{A}$ and $\tilde{B}$ such that each pair of points $x, y \in G$ can be joined by a rectifiable arc $\gamma \subset G$ for which

$$\begin{cases} s(\gamma) \leq \tilde{A}|x - y|, \\ \min\{s(\gamma_x), s(\gamma_y)\} \leq \tilde{B}d(z) \quad \text{for all } z \in \gamma, \end{cases}$$

where $s(\gamma)$ denotes the Euclidean length of γ and γ_x (γ_y) denotes the part of γ between z and x (resp. z and y).

Next define $j_G(x, y) = \log(1 + |x - y|/\min\{d(x), d(y)\})$ for $x, y \in G$.

The following characterization of uniform domains is due to Gehring and Osgood. Because our definition of $j_G(x, y)$ differs slightly from the corresponding definition in [2] the numerical values of the costants c and d may be slightly different from those in [2].

2.1. LEMMA. *A domain G is uniform if and only if there are constants c and d such that $k_G(x, y) \leq c j_G(x, y) + d$ for all $x, y \in G$.*

2.2. REMARK. If G is uniform, then $k_G(x, y) \leq B j_G(x, y)$ for all $x, y \in G$ where $B \leq 3(c + d)$ as shown in [7]. A similar observation has been independently made by K. Astala and F. W. Gehring (unpublished).

For the theory of quasiconformal mappings and for the properties of the modulus of a curve family the reader is referred to Väisälä's book [6]. For $x \in R^n$ and $r > 0$ let $B^n(x, r) = \{z \in R^n : |z - x| < r\}$ and $S^{n-1}(x, r) = \partial B^n(x, r)$.

3. The result. We first prove a lemma. The boundary of a domain in R^n is taken in the topology of $\overline{R}^n = R^n \cup \{\infty\}$.

3.1. LEMMA. *Let G be a domain in R^n with connected boundary ∂G, let $a, b \in G$, and let $x \in \partial G$ be such that $d(a) = |a - x| \leq |b - x|$. For every homeomorphism $f : \overline{G} \to \overline{G}$ with $f(a) = b$ and $f|\partial G = id|\partial G$, the following lower bound holds*

$$K(f) \geq d_n \left(\log \frac{|b - x|}{|a - x|} \right)^n,$$

where d_n is a positive number depending only on n.

PROOF. Write $m = |a - x|$ and $M = |b - x|$. Then the segment $[a, x] = \{au + x(1 - u) : 0 \leq u \leq 1\} \subset G \cup \{x\}$. We may assume $M > m$. Fix $t \in (m, M)$. Write $A = \partial G \setminus \overline{B}^n(x, t)$ and let $\Gamma_t = \Delta([a, x], A; G)$ be the family of all curves in G joining $[a, x]$ to A [6, p. 21]. Since $f|\partial G = id|\partial G$ it follows that $fA = A$ and further because $f(a) = b$ and ∂G is connected

(3.2) $S^{n-1}(x, r) \cap f[a, x] \neq \varnothing, \qquad S^{n-1}(x, r) \cap fA \neq \varnothing,$

for $r \in (t, M)$. From (3.2) and [6, 7.5, 10.12, 13.1] it follows that

$$c_n \log \frac{M}{t} \leq M(f\Gamma_t) \leq K(f)M(\Gamma_t) \leq K(f)\omega_{n-1} \left(\log \frac{t}{m} \right)^{1-n},$$

where c_n and ω_{n-1} are positive numbers depending only on n. Since $t \in (m, M)$ is arbitrary, we get

$$K(f) \geq \frac{c_n}{\omega_{n-1}} \sup\{h(t) \colon t \in [m, M]\}; \qquad h(t) = \left(\log \frac{M}{t}\right)\left(\log \frac{t}{m}\right)^{n-1}.$$

Set $t = m^p M^{1-p}$, $p \in (0, 1)$. Then

$$h(m^p M^{1-p}) = p(1 - p)^{n-1}\left(\log \frac{M}{m}\right)^n.$$

For $p = 1/n$ we get the desired lower bound

$$K(f) \geq d_n \left(\log \frac{M}{m}\right)^n; \qquad d_n = c_n(n - 1)^{n-1}/(\omega_{n-1} n^n).$$

3.3. THEOREM. *Let G be a uniform domain in R^n with connected boundary ∂G, and let $a, b \in G$. Then there exists a positive constant $c(n, G)$ depending only on n and G such that for every homeomorphism $f \colon \overline{G} \to \overline{G}$ with $f(a) = b$ and $f|\partial G = id|\partial G$*

$$K(f) \geq c(n, G)k_G(a, b)^n$$

whenever $k_G(a, b)$ exceeds a bound, depending only on n and G.

PROOF. We may assume $d(a) \leq d(b)$. Fix $z \in \partial G$ such that $d(a) = |a - z|$. Then we have

$$(3.4) \qquad \frac{|b - z|}{|a - z|} = \frac{|b - z|}{d(a)} > \frac{|a - b|}{|a - z| + |b - z|} \frac{|b - z|}{d(a)} \geq \frac{1}{2}\frac{|a - b|}{d(a)}.$$

Since G is uniform, there exists a number B such that $k_G(x, y) \leq Bj_G(x, y)$ for all $x, y \in G$ (cf. Remark 2.2). Since $d(a) \leq d(b)$, we get by the definition of j_G

$$\frac{|a - b|}{d(a)} \geq \exp(k_G(a, b)/B) - 1 \geq k_G(a, b)/B.$$

Assume now that $k_G(a, b) \geq 2B$. For $k_G(a, b) \geq 2B$ we can improve the last inequality as follows

$$\frac{|a - b|}{d(a)} \geq \exp(k_G(a, b)/B) - 1 \geq 2\exp(k_G(a, b)/(2B)).$$

This inequality, together with 3.1 and (3.4) yields

$$K(f) \geq c(n, G)k_G(a, b)^n; \quad c(n, G) = d_n(2B)^{-n}$$

for $k_G(a, b) \geq 2B$, as desired.

3.5. REMARK. It is an interesting question, whether the best possible lower bound for $K(f)$ in 3.3 is greater than one whenever $a \neq b$. We now show that this is not true when $n = 3$ and $G = R^3\backslash Z$, $Z = \{(0, 0, z)\colon z \in R\}$. Let $a = (1, 0, 0)$ and $b = -a$, and let f be a rotation around the z-axis with $f(a) = b$ such that f keeps the z-axis Z pointwise fixed. Then f is conformal and hence $K(f) = 1$, while by a symmetry argument we see that $k_G(a, b) = \pi$. It is easy to verify that $R^3\backslash Z$ is uniform.

REFERENCES

1. G. D. Anderson and M. K. Vamanamurthy, *An extremal displacement mapping in n-space*, Complex analysis, Joensuu, 1978, pp. 1–9; Lecture Notes in Math., vol. 747, Springer-Verlag, 1979.
2. F. W. Gehring and B. G. Osgood, *Uniform domains and the quasihyperbolic metric*, J. Analyse Math. **36** (1979), 50–74.
3. F. W. Gehring and B. P. Palka, *Quasiconformally homogeneous domains*, J. Analyse Math. **30** (1976), 172–199.
4. O. Martio and J. Sarvas, *Injectivity theorems in plane and space*, Ann. Acad. Sci. Fenn. Ser. A I Math. **4** (1978/1979), 383–402.
5. O. Teichmüller, *Ein Verschiebungssatz der quasikonformen Abbildung*, Deutsche Math. **7** (1944), 336–343.
6. J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Math., vol. 229, Springer-Verlag, 1971.
7. M. Vuorinen, *Conformal invariants and quasiregular mappings*, in preparation.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HELSINKI, HALLITUSKATU 15, SF-00100 HELSINKI 10, FINLAND