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CONVEX FUNCTIONS OF BOUNDED TYPE

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ABSTRACT. We introduce a new class of normalized functions univalent and convex in the unit disk. These are called convex of bounded type and the set is denoted by $CV(R_1, R_2)$. For this set we find the Koebe domain, a coefficient bound, and a bound for $|f(z)|$. We also mention a few of the many questions that can be asked about this new class of univalent functions.

1. Introduction. Let $CV(\alpha)$ denote the set of functions that are convex of order α. These are the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are regular and univalent in the unit disk E: $|z| < 1$, and for which

$$\operatorname{Re} Q_{CV}(f) \equiv \operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) > \alpha, \qquad z \text{ in } E. \tag{1.2}$$

Here, of course, we must have $0 \leqslant \alpha \leqslant 1$ (see [**3** and **1**, vol. I, pp. 137–142]).

The class $CV(\alpha)$ has been studied extensively, but the geometric properties of $f(E)$ under a function in $CV(\alpha)$ are not immediately clear. The condition (1.2) states that the curve C that bounds $f(E)$ satisfies the condition

$$d\psi/d\theta \geqslant \alpha \tag{1.3}$$

whenever this derivative exists. The difficulty lies in the fact that ψ is an angle in the w-plane and θ is an angle in the z-plane. Thus the geometric implication of (1.3) is not obvious. Our purpose is to introduce and study a similar class of functions where the geometric nature of $f(E)$ is readily observable. Briefly we place upper and lower bounds on the curvature of C. It is more convenient to use ρ, the radius of curvature (the reciprocal of the curvature). By a formula due to Study [**4**], the radius of curvature of $f(|z| = r)$ is given by

$$\rho = \frac{|zf'(z)|}{\operatorname{Re} Q_{CV}(f)}, \qquad z = re^{i\theta}. \tag{1.4}$$

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Briefly we ask that $R_1 \leqslant \rho \leqslant R_2$. However, such a condition cannot be imposed throughout E because $\rho \to 0$ as $z \to 0$, and our interest centers on the boundary of E. This forces a more complicated definition. Let

$$\rho_1(r) = \min_{|z|=r} \rho \quad \text{and} \quad \rho_2(r) = \max_{|z|=r} \rho. \tag{1.5}$$

Set

$$R_3 = \liminf \rho_1 \quad \text{and} \quad R_4 = \limsup \rho_2 \tag{1.6}$$

as $r \to 1^-$.

DEFINITION 1. *Let R_1 and R_2 be fixed in* $[0, \infty]$. *We say that $f(z)$ of the form* (1), *regular and univalent in E, is in the class $CV(R_1, R_2)$ if $R_1 \leqslant R_3$ and $R_4 \leqslant R_2$. A function in $CV(R_1, R_2)$ with $0 < R_1 \leqslant R_2 < \infty$ is said to be a convex function of bounded type.*

Thus, by definition, the sets $CV(R_1, R_2)$ are increasing as either $R_1 \to 0$ or $R_2 \to \infty$, and the union over all R_1, R_2 is the set of all normalized convex functions. Let C be the boundary of $f(E)$. Then by definition, if $f(z) \in CV(R_1, R_2)$, then on C

$$R_1 \leqslant ds/d\psi = \rho \leqslant R_2. \tag{1.7}$$

Here s is arc length on C and ψ is the angle the tangent to C makes with the positive real axis. Since both s and ψ are in the w-plane, the geometric character of $f(z)$ is clear.

If a simple closed curve satisfies the condition (1.7) with $0 < R_1 \leqslant R_2 < \infty$, we will call it a convex curve of bounded type, and (by abuse of notation) we will write that $C \in CV(R_1, R_2)$. The investigation of such curves has a long history. Some useful results and further references may be found in [**2**].

Clearly the set of functions $CV(R_1, R_2)$ is invariant under the rotation $g(z) = e^{-i\gamma}f(ze^{i\gamma})$. Let $\overline{CV}(R_1, R_2)$ be the subset of $CV(R_1, R_2)$ for which the bounds R_1 and R_2 in (1.7) are actually attained on C. The transformation

$$g(z) = \frac{f((z+a)/(1+\bar{a}z)) - f(a)}{f'(z)(1-|a|^2)} \tag{1.8}$$

will take a function in $\overline{CV}(R_1, R_2)$ into a function in the same class, if and only if $|f'(a)(1-|a|^2)| = 1$. Thus (1.8) seems to be useless in the study of $CV(R_1, R_2)$.

In many studies of the set S and its various subsets, the new function $g(z) \equiv f(rz)/r$, with $0 < r < 1$, will belong to the same set as the primitive function $f(z)$ does. This pleasant property permits the author to prove a theorem about $g(z)$ which is analytic on $|z| = 1$, and by taking the limit as $r \to 1^-$, obtain the same result about functions $f(z)$ in the same set when $f(z)$ is not analytic on $|z| = 1$.

Unfortunately, the set $CV(R_1, R_2)$ does not behave quite as desired. If $f(z) \in CV(R_1, R_2)$ and r is fixed in $(0, 1)$ it is possible that $g(z) \equiv f(rz)/r$ is not in $CV(R_1, R_2)$. However, one can prove that as $r \to 1^-$, the change in R_1 and R_2 is negligible. Thus (omitting a few details) we can always prove a theorem about $C_r = f(|z| = r)$ and then take the limit as $r \to 1^-$. Hence, without loss of generality, we may assume that $f(z)$ is analytic in $|z| \leqslant 1$.

2. A coefficient bound. The function

$$F(z) \equiv \frac{z}{1 - Az} = z + \sum_{n=2}^{\infty} A^{n-1}z^n, \qquad 0 \leqslant A < 1, \tag{2.1}$$

maps E conformally onto the disk with center $A/(1 - A^2)$ and radius $1/(1 - A^2)$. Hence $F(z) \in CV(R_1, R_2)$ where $R_2 = 1/(1 - A^2)$ and R_1 is any number in $(0, R_2]$. On the other hand, if $f(z) \in CV(R_1, R_2)$ then $f(E)$ is contained in some disk of radius R_2 (see [**2**]). An area theorem [**1**, vol. I, p. 27] gives

THEOREM 1. *If $f(z) \in CV(R_1, R_2)$, then*

$$1 + \sum_{n=2}^{\infty} n|a_n|^2 \leqslant R_2^2, \tag{2.2}$$

and for each $k \geqslant 2$,

$$|a_k| \leqslant \left(\left(R_2^2 - 1\right)/k\right)^{1/2} < R_2/k^{1/2}. \tag{2.3}$$

The first inequality is sharp for each pair with $0 \leqslant R_1 \leqslant R_2$.

From (2.2) we see that $R_2 \geqslant 1$, and the set $CV(R_1, 1)$ contains only one member, $f(z) \equiv z$. The example function (2.1) suggests the conjecture that for all k and R_2 and $f(z)$ in $CV(R_1, R_2)$,

$$|a_k| \leqslant A^{k-1} \equiv (1 - 1/R_2)^{(k-1)/2}, \qquad R_2 \geqslant 1. \tag{2.4}$$

If (2.4) were true, it would be sharp, and thus a great improvement over (2.3). Now (2.4) may be the true bound for some values of k and R_2, but the following example shows that (2.4) cannot be correct for all $k \geqslant 2$ and $R_2 > 1$.

Set

$$G(z) = z + az^k, \qquad a \geqslant 0, k \geqslant 2. \tag{2.5}$$

It is well known that $G(z)$ is convex if and only if $0 \leqslant ak^2 \leqslant 1$. A moderate computation shows that $G(z) \in CV(R_1, R_2)$ where

$$R_2 = (1 - ka)^2/(1 - k^2a), \qquad 0 \leqslant ak^2 < 1. \tag{2.6}$$

The value of R_1 is not needed in what follows. For small values of k we find that $a \leqslant (1 - 1/R_2)^{(k-1)/2}$. However if we set $a = 1/1000$ and $k = 17$ in (2.6), we find that $R_2 \approx 1.359$. On the other hand, these values used in (2.4) give $A^{k-1} \approx 0.0000237 < 1/1000$. Hence (2.4) cannot give the sharp bound for $k = 17$ and $R_2 \approx 1.359$.

3. Koebe domains. Let $d = |w_0|$ where w_0 is a point nearest the origin on $\partial f(E)$.

THEOREM 2. *If $f(z) \in CV(R_1, R_2)$, then*

$$|f(z)| \leqslant 2R_2 - d, \qquad z \in \bar{E}. \tag{3.1}$$

Further,

$$d \geqslant R_2 - \left(R_2^2 - R_2\right)^{1/2} \equiv R_K, \tag{3.2}$$

and hence $f(E)$ always covers the disk centered at the origin with radius R_K. Both inequalities are sharp.

PROOF. By a rotation, we may set $w_0 = -d$. The line from the origin to w_0 is normal to $\partial f(E)$ at w_0. From **[2]** a disk of radius R_2 and center at $R_2 - d$ will cover $f(E)$. This proves (3.1). Further $F(z)$ given by (2.1) shows that for each $R_2 \geqslant 1$, the inequality (3.1) is sharp. For this function, $R_2 = 1/(1 - A^2)$ and d is given by (3.2) with the equal sign.

Since the disk described above covers $f(E)$,

$$f(z) \prec \frac{Bz}{1 - Az} \tag{3.3}$$

where $B = (2R_2 - d)d/R_2$ and $A = (R_2 - d)/R_2$. But $f'(0) = 1$, and hence $B \geqslant 1$, with equality if and only if $f(z) = z/(1 - Az)$. A brief computation with $(2R_2 - d)d/R_2 \geqslant 1$ will give (3.2). This same function $z/(1 - Az)$ with suitable A, shows that (3.2) is sharp. By a rotation, the inequality (3.2) gives the disk $|z| < R_K$ as the Koebe domain for the set $CV(R_1, R_2)$ for each $R_2 \geqslant 1$. □

The inequalities (3.1) and (3.2) give the

COROLLARY. *If $f(z) \in CV(R_1, R_2)$ and $1/2 \leqslant d \leqslant 1$, then*

$$R_2 \geqslant \frac{d^2}{2d - 1} \geqslant 1 \tag{3.4}$$

and

$$|f(z)| \leqslant R_2 + \left(R_2^2 - R_2\right)^{1/2}, \quad z \text{ in } E. \tag{3.5}$$

Both inequalities are sharp.

We next consider a subordination in the reverse direction of (3.3). If w_0 is a point of $\partial f(E)$ that is closest to the origin, we may set $w_0 = -d$ by a suitable rotation. From **[2]** the domain $f(E)$ will contain the open disk with radius R_1 and center $R_1 - d$. Then $Bz/(1 - Az) \prec f(z)$ where $B = (2R_1 - d)d/R_1$ and $A = (R_1 - d)/R_1$. The condition $B \leqslant 1$ will give

THEOREM 3. *If $f(z) \in CV(R_1, R_2)$ and $R_1 \geqslant 1$, then $d \leqslant R_1 - (R_1^2 - R_1)^{1/2}$ and*

$$R_1 \leqslant \frac{d^2}{2d - 1}, \qquad \frac{1}{2} < d \leqslant 1.$$

Both inequalities are sharp.

Together with (3.2) we have

$$R_2 - \left(R_2^2 - R_2\right)^{1/2} \leqslant d \leqslant R_1 - \left(R_1^2 - R_1\right)^{1/2} \tag{3.6}$$

when $R_1 \geqslant 1$.

4. Convex functions of order α. Does either of the sets $CV(R_1, R_2)$ and $CV(\alpha)$ contain the other?

THEOREM 4. *If $R_2 < \infty$, and $0 \leqslant \alpha < 1$, then*

$$CV(\alpha) \not\subset CV(R_1, R_2). \tag{4.1}$$

PROOF. The function

$$f(z) = \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1} \tag{4.2}$$

is in $CV(\alpha)$ if $\alpha \neq 1/2$ and $0 \leqslant \alpha \leqslant 1$. The function

$$f(z) = -\ln(1-z) \tag{4.3}$$

is in $CV(1/2)$. If $0 \leqslant \alpha \leqslant 1/2$, then the above examples are unbounded in E and hence cannot belong to $CV(R_1, R_2)$ for any finite R_2 (see Theorem 2).

If $1/2 < \alpha < 1$, and $z = e^{i\theta}$, a brief computation, using (1.4), gives

$$\rho = 2^{\alpha}/2\alpha(1-\cos\theta)^{1-\alpha}. \tag{4.4}$$

Hence $\rho \to \infty$ as $\theta \to 0$. □

THEOREM 5. *If $f(z) \in CV(R_1, R_2)$ and $R_2 < \infty$, then for some α,*

$$CV(R_1, R_2) \subset CV(\alpha). \tag{4.5}$$

In fact, $\alpha > 1/4R_2$.

PROOF. From (1.4) and the definition of $CV(R_1, R_2)$ it follows that on the boundary of E

$$\frac{\min|zf'(z)|}{\min \operatorname{Re} Q_{CV}(f)} \leqslant R_2 \tag{4.6}$$

or

$$\operatorname{Re} Q_{CV}(f) \geqslant \frac{\min|zf'(z)|}{R_2}, \qquad z = e^{i\theta}. \tag{4.7}$$

Since the left side of (4.7) is a harmonic function, a minimum on ∂E will hold throughout E. It is well known that if $f(z) \in CV(\alpha)$, then

$$|f'(z)| \geqslant 1/(1+r)^{2(1-\alpha)} \geqslant 4^{-(1-\alpha)}. \tag{4.8}$$

Since a function in $CV(R_1, R_2)$ is also convex, we can use (4.8) with $\alpha = 0$. Then (4.7) gives $\operatorname{Re} Q_{CV}(f) \geqslant 1/4R_2$. □

This procedure can be iterated. Now that $f(z)$ is in $CV(\alpha)$ with $\alpha = \alpha_1 = 1/4R_2$, we can use this in (4.8) and (4.7) to generate an α_2. In general, the sequence

$$\alpha_{k+1} = \frac{1}{(4^{1-\alpha_k})R_2} \tag{4.9}$$

is a bounded increasing sequence that has a limit β. Then (4.5) holds with $\alpha \geqslant \beta$. However, it is clear that β is not the best lower bound for α and hence the precise determination of β as the root of $\beta = 1/4^{(1-\beta)}R_2$ that lies in $(0, 1)$ is not important. Using a series for 4^{-1+x} we can show that

$$\alpha > \beta > \frac{1}{4R_2} + \frac{\ln 4}{16R_2^2}.$$

With a suitable choice of A, the function $z/(1-Az)$ is an example that lies in $CV(R_1, R_2)$ and in $CV(\gamma)$, where

$$\gamma = 2R_2 - 1 - 2\left(R_2^2 - R_2\right)^{1/2} = \frac{1}{4R_2} + \frac{1}{8R_2^2} + \cdots. \tag{4.10}$$

It is reasonable to conjecture that γ is the sharp (largest) value of α for which (4.5) is true.

5. Other questions. What are the sharp bounds for $|a_k|$? It seems as though variational formulas for other classes of univalent functions cannot be applied to the class $\overline{CV}(R_1, R_2)$ or $CV(R_1, R_2)$. Is there a "nice" variational formula for either of these two classes?

We can obtain an Alexander type theorem if we define a class $ST(R_1, R_2)$ of starlike functions of bounded type. Thus $F(z) \in ST(R_1, R_2)$ if and only if $F(z) = zf'(z)$ for some $f(z)$ in $CV(R_1, R_2)$. For such a function

$$R_1 \leqslant \frac{|F(z)|}{\operatorname{Re}(zF'(z)/F(z))} \leqslant R_2 \tag{5.1}$$

as $|z| \to 1$. Then each theorem about $CV(R_1, R_2)$ will yield a companion theorem for the class $ST(R_1, R_2)$. But what are the geometric properties of functions in $ST(R_1, R_2)$?

Finally, we might ask questions about a new class of normalized functions $F(z)$ for which

$$R_1 \leqslant \frac{|zF'(z)|}{\operatorname{Re}(zF'(z)/F(z))} \leqslant R_2 \tag{5.2}$$

as $|z| \to 1$. For such functions $R_1 \leqslant ds/d\Phi \leqslant R_2$ on $\partial F(E)$ where $\Phi = \arg F(e^{i\theta})$ and s is arc length. Here the geometric character of $F(E)$ is clear, but the relation of this class to the classes $CV(R_1, R_2)$ and $ST(R_1, R_2)$ is not.

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