ON A THEOREM OF INGHAM
ON NONHARMONIC FOURIER SERIES

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ABSTRACT. A well-known result due to Ingham [3] shows that the system
of complex exponentials $\{e^{i\lambda_n t}\}$ is a basic sequence in $L^2(-\pi, \pi)$ whenever
$\lambda_{n+1} - \lambda_n \geq \gamma > 1$. In this note, we show that the system need not be basic
if $\lambda_{n+1} - \lambda_n > 1$.

1. Introduction. Let $\{\lambda_n\}$ be an increasing sequence of real numbers. A well-known result due to Ingham [3] states that if $\lambda_{n+1} - \lambda_n \geq \gamma > 1$ then the series $\sum c_n e^{i\lambda_n t}$ converges in $L^2(-\pi, \pi)$ whenever $\sum |c_n|^2 < \infty$ and, moreover,

$$(1) \qquad A \sum |c_n|^2 \leq \left\| \sum c_n e^{i\lambda_n t} \right\|^2 \leq B \sum |c_n|^2 .$$

Here, A and B are positive constants depending only on γ. (That the right-hand inequality is valid whenever $\gamma > 0$ appears to have been first proved by Titchmarsh [10].)

Ingham showed that his result is the best possible in the sense that if $\gamma = 1$ then the left-hand inequality cannot obtain. A counterexample is provided by the sequence $\{\lambda_n\}$ where

$$(2) \qquad \lambda_n = \begin{cases} n - 1/4, & n > 0, \\ n + 1/4, & n < 0. \end{cases}$$

It follows readily from (1) that the system of exponentials $\{e^{i\lambda_n t}\}$ is a *basic sequence* in $L^2(-\pi, \pi)$, that is, a basis for its closed linear span S. Accordingly, each function f in S has a unique representation

$$f(t) = \sum c_n e^{i\lambda_n t} \quad \text{(in the mean)}.$$

The study of such *nonharmonic Fourier series* was initiated by Paley and Wiener [5] who showed that the system $\{e^{i\lambda_n t}\}$ is a basis for $L^2(-\pi, \pi)$ whenever the λ_n are sufficiently close to the integers. Since then the theory has been generalized in many ways and in many different directions (see, e.g., [2, 6, 8, 11] and the references therein).

Condition (1), while tractable, is a stringent requirement to place on a basic sequence. Nevertheless, we show in Theorem 1 that the right-hand inequality must obtain for every basic sequence of exponentials. At present, there is no known example of such a sequence for which the left-hand inequality does not also obtain. Theorem 2 further dramatizes the strength of Ingham's result by showing that the slightly weaker separation condition $\lambda_{n+1} - \lambda_n > 1$ cannot even guarantee that the system $\{e^{i\lambda_n t}\}$ is a basic sequence in $L^2(-\pi, \pi)$.

THEOREM 1. *If* $\{\lambda_n\}$ *is an increasing sequence of real numbers for which the system of exponentials* $\{e^{i\lambda_n t}\}$ *is a basic sequence in* $L^2(-\pi, \pi)$, *then the inequality* $\|\sum c_n e^{i\lambda_n t}\|^2 \leq B \sum |c_n|^2$ *is valid for some constant* B *and all square summable sequences of scalars* $\{c_n\}$.

THEOREM 2. *There exists a sequence* $\{\mu_n\}$ *of real numbers satisfying* $\mu_{n+1} - \mu_n > 1$ *such that* $\{e^{i\mu_n t}\}$ *is exact in* $L^2(-\pi, \pi)$ *and yet not a basis.*

Recall that $\{e^{i\mu_n t}\}$ is said to be *exact* if it is complete but fails to be complete upon the removal of a single term.

2. Proof of Theorem 1. We need only show that the λ_n are separated, i.e., that $\lambda_{n+1} - \lambda_n \geq \gamma$ for some positive constant γ; the result will then follow from [10].

Let S be the closure in $L^2(-\pi, \pi)$ of the linear span of the system $\{e^{i\lambda_n t}\}$, and let $\{f_n\}$, $f_n \in S^*$, be the associated sequence of coefficient functionals. Then $\|e^{i\lambda_n t}\| \, \|f_n\| \leq M$ for some constant M and all values of n (see, e.g., [9, p. 20]). Since each λ_n is real, $\|e^{i\lambda_n t}\| = 1$ and hence $\|f_n\| \leq M$. Now $f_n(e^{i\lambda_n t} - e^{i\lambda_{n+1} t}) = 1$ so that $\|f_n\| \, \|e^{i\lambda_n t} - e^{i\lambda_{n+1} t}\| \geq 1$. Accordingly, $\|e^{i\lambda_n t} - e^{i\lambda_{n+1} t}\| \geq 1/M$ and the existence of γ follows.

3. Proof of Theorem 2. The system $\{e^{i\lambda_n t}\}$ where the λ_n are given by (1), is known to be exact in $L^2(-\pi, \pi)$ [4, p. 67]. We begin by showing that it is not a basis. Suppose it were. Then we could write

$$(3) \qquad\qquad 1 = \sum c_n e^{i\lambda_n t} \quad \text{(in the mean)}.$$

To compute the c_n, we shall make use of the Paley-Wiener space P consisting of all entire functions of exponential type at most π that are square integrable on the real axis. The inner product of two functions F and G in P is, by definition,

$$(F, G) = \int_{-\infty}^{\infty} F(x)\overline{G(x)} \, dx.$$

By virtue of the Paley-Wiener theorem, the complex Fourier transform

$$f(t) \rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{izt} \, dt$$

is an isometric isomorphism from $L^2(-\pi, \pi)$ onto all of P. The exponentials $e^{i\lambda_n t}$ are sent to the "reproducing" functions

$$K_n(z) = \frac{\sin \pi(z - \lambda_n)}{\pi(z - \lambda_n)}$$

which then consistute a basis for P. Let $\{g_n\}$ be biorthogonal to $\{K_n\}$. When the Fourier transform is applied to (3), we obtain

$$\frac{\sin \pi z}{\pi z} = \sum c_n K_n(z)$$

where $c_n = ((\sin \pi z)/\pi z, g_n) = g_n(0)$.

Let

$$F(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right).$$

It was shown by Levinson [**4**, p. 67] that

$$F(z) = c \int_{-\pi}^{\pi} \left(\cos^{-1/2} \frac{1}{2} t \right) e^{izt} \, dt.$$

Since $\cos^{-1/2} \frac{1}{2} t$ is integrable over $(-\pi, \pi)$, it follows that $F(z)$ is bounded along the real axis and each of the functions

$$F_n(z) = F(z)/F'(\lambda_n)(z - \lambda_n)$$

therefore belongs to P. Since $(F_n, K_m) = F_n(\lambda_m) = \delta_{mn}$, it follows that $F_n = g_n$. Thus

$$c_n = F_n(0) = -1/\lambda_n F'(\lambda_n),$$

and (3) becomes

$$(4) \qquad 1 = -\sum_{n \neq 0} \frac{e^{i\lambda_n t}}{\lambda_n F'(\lambda_n)} = -2 \sum_{n=1}^{\infty} \frac{\cos \lambda_n t}{\lambda_n F'(\lambda_n)}$$

since $zF'(z)$ is even. It is to be shown that the series on the right does not converge in $L^2(-\pi, \pi)$.

Now the values of $F'(\lambda_n)$ were determined explicitly in [**7**]:

$$F'(\lambda_n) = (-1)^n \Gamma^2 \left(\frac{3}{4} \right) \frac{\Gamma(n)}{\Gamma\left(n + \frac{1}{2}\right)} \qquad (n = 1, 2, 3, \dots).$$

Using the asymptotic formula $\Gamma(n)/\Gamma(n + \frac{1}{2}) = 1/\sqrt{n} + O(1/n^{3/2})$ [**1**], we have

$$F'(\lambda_n) = A(-1)^n \left\{ 1/\sqrt{\lambda_n} + \varepsilon_n \right\} \quad \text{where } \varepsilon_n = O(1/n^{3/2}).$$

A straightforward calculation then shows that the difference between the series on the right in (4) and

$$\frac{1}{A} \sum_{n=1}^{\infty} \frac{(-1)^n \cos \lambda_n t}{\sqrt{\lambda_n}}$$

is uniformly convergent on $[-\pi, \pi]$. Accordingly, we need only show that this series diverges in $L^2(-\pi, \pi)$.

Let $x = \pi - t$ $(0 \leq t \leq \pi)$. Then

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cos \lambda_n t}{\sqrt{\lambda_n}} = \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\pi n - \pi/4 - \lambda_n x)}{\sqrt{\lambda_n}}$$

$$= \sum_{n=1}^{\infty} \frac{\cos(\pi/4 + \lambda_n x)}{\sqrt{\lambda_n}}.$$

For $N = 1, 2, 3, \dots$, let $\delta_N = \pi/16N$. If $x \in [0, \delta_N]$, then $\pi/4 + \lambda_n x \in [\pi/4, 3\pi/8]$ whenever $1 \leq n \leq 2N$ and hence $\cos(\pi/4 + \lambda_n x) \geq A > 0$. Thus

$$\sum_{N}^{2N} \frac{\cos(\pi/4 + \lambda_n x)}{\sqrt{\lambda_n}} \geq A \sum_{N}^{2N} \frac{1}{\sqrt{n - 1/4}} \geq A \frac{N+1}{\sqrt{2N - 1/4}} \geq B\sqrt{N}$$

where B is a positive constant independent of N. Accordingly,

$$\left\| \sum_{N}^{2N} \frac{(-1)^n \cos \lambda_n t}{\sqrt{\lambda_n}} \right\|^2 \geq \left\| \sum_{N}^{2N} \frac{\cos(\pi/4 + \lambda_n x)}{\sqrt{\lambda_n}} \right\|^2_{L^2(0,\delta_N)}$$

$$\geq \frac{1}{2\pi} B^2 N \delta_N = \frac{B^2}{32}$$

for all N. Thus the series in (3) does not converge in $L^2(-\pi, \pi)$, and the system $\{e^{i\lambda_n t}\}$ fails to be a basis.

Let $\{\varepsilon_1, \varepsilon_2, \ldots\}$ be a decreasing sequence of positive numbers such that $\varepsilon_1 < \frac{1}{4}$ and $\sum \varepsilon_n < \infty$. It is to be shown that if $\mu_n = \lambda_n - \varepsilon_n$, $\mu_{-n} = -\mu_n$ $(n = 1, 2, \ldots)$, then the system $\{e^{i\mu_n t}\}$ satisfies the conclusions of the theorem.

Clearly, $\mu_{n+1} - \mu_n > 1$. Since $\sum |\lambda_n - \mu_n| < \infty$ and $\{e^{i\lambda_n t}\}$ is exact, so is $\{e^{i\mu_n t}\}$ [6]. It remains only to show that $\{e^{i\mu_n t}\}$ is not a basis for $L^2(-\pi, \pi)$. Suppose it were. Then the system $\{h_n(t)\}$, biorthogonal to $\{e^{i\mu_n t}\}$, would satisfy $\|e^{i\mu_n t}\| \cdot \|h_n\| \leq M$ for some constant M and all values of n. Since each μ_n is real, $\|e^{i\mu_n t}\| = 1$ and hence $\|h_n\| \leq M$. We complete the proof by showing that

$$(5) \qquad \sum |\lambda_n - \mu_n| < \infty \Rightarrow \sum \|e^{i\lambda_n t} - e^{i\mu_n t}\| < \infty.$$

The convergence of $\sum \|e^{i\lambda_n t} - e^{i\mu_n t}\| \, \|h_n\|$ will then imply that $\{e^{i\lambda_n t}\}$ is a basis for $L^2(-\pi, \pi)$ (see, e.g., [9, p. 94]). The contradiction will prove the theorem.

To establish (5), write

$$e^{i\lambda_n t} - e^{i\mu_n t} = e^{i\lambda_n t}\left(1 - e^{-i\varepsilon_n t}\right).$$

Expanding $1 - e^{i\gamma t}$ in an everywhere-convergent Taylor series, we find

$$|e^{i\lambda_n t} - e^{i\mu_n t}| \leq \sum_{K=1}^{\infty} \frac{\varepsilon_n^K t^K}{K!},$$

and hence

$$\sum_n \|e^{i\lambda_n t} - e^{i\mu_n t}\| \leq \sum_{K=1}^{\infty} \frac{\pi^K}{K!} \left(\sum_n \varepsilon_n^K \right) \leq \sum_{K=1}^{\infty} \frac{\pi^K}{K!} \left(\sum \varepsilon_n \right)^K$$

$$= \exp\left(\pi \sum \varepsilon_n\right) - 1 < \infty.$$

This completes the proof.

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