DERIVATIVES OF BERNSTEIN POLYNOMIALS AND SMOOTHNESS

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ABSTRACT. Equivalence relations between the asymptotic behaviour of derivatives of Bernstein polynomials and the smoothness of the function they approximate are given. This is achieved with an a priori condition that the function is of class $\text{Lip } \beta$ with some small $\beta > 0$. The a priori condition is dropped when a similar equivalence relation using the Katorovich operator is proved.

1. Introduction. In many articles the rate of convergence of the Bernstein polynomial $B_n(f, x)$ to $f(x)$ is related to the smoothness of f (for example, see [1–5]). Sometimes it is possible to find information about the smoothness of f from derivatives of the approximation process, as is well known in the case $P_y * f$, where P_y is the Poisson kernel and was investigated in many other approximation processes given by convolutions. In this paper the relation will be investigated for the special but important operator of Bernstein polynomials that obviously is not a convolution. We will show that $|\Delta_h^2 f| \leq Mh^\alpha$ implies

$$|B_n''(f, t)| \leq M_1 \{ \min(n^2, n/t(1-t)) \}^{1-\alpha/2}$$

and the latter together with $|\Delta_h^2 f| \leq M_2 h^\beta$ (no matter how small β is) implies $|\Delta_h^2 f| \leq Mh^\alpha$. The situation will be shown to be better for the Kantorovich operator.

2. The main result. The Bernstein polynomials are given by

$$(2.1) \quad B_n(f, x) \equiv \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) \equiv \sum_{k=0}^n f\left(\frac{k}{n}\right) P_{k,n}(x).$$

Recalling $\Delta_h f(x) = f(x+h) - f(x)$ and $\Delta'_h f(x) = \Delta_h(\Delta_h^{-1} f(x))$, the moduli of continuity are given by

$$(2.2) \quad \omega_r(f, \eta, [a, b]) = \text{Sup} \{ |\Delta'_h f(x)|; h \leq \eta, [x, x+rh] \subset [a, b] \}.$$

We can now state our result.

THEOREM 2.1. *For $f \in C[0, 1] \equiv C(I)$ and for some $\beta > 0$ and $\omega_1(f, \eta, I) \leq K\eta^\beta$ we have the following equivalence relations:*

(a) *For $\alpha \leq 1$, $\omega_1(f, h, I) \leq Mh^\alpha$ if and only if*

$$|B_n'(f, t)| \leq M_1 \{ \min(n^2, n/t(1-t)) \}^{(1-\alpha)/2}.$$

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(b) For $\alpha \leq 2$, $\omega_2(f, h, I) \leq Mh^\alpha$ if and only if

$$|B_n''(f, t)| \leq M_1 \left\{ \min(n^2, n/t(1-t)) \right\}^{1-\alpha/2}.$$

REMARK. Obviously if $\omega_i(f, h, I) \leq Mh^\alpha$ ($i \geq 1$), the condition $|\Delta_h f(x)| \leq Kh^\beta$ is redundant. We will later show that in some cases this additional side condition can be removed. It is an open question whether it can always be removed.

For our proof we will need the following lemma:

LEMMA 2.2. For $f \in C[0, 1]$ there exist $f_{1,\eta}$ and $f_{2,\eta}$ such that

$$|f(x) - f_{1,\eta}(x)| \leq \omega_1(f, \eta/2), \quad |f(x) - f_{2,\eta}(x)| \leq 5\omega_2(f, \eta),$$

$$|f'_{1,\eta}(x)| \leq \frac{1}{\eta} \omega_1(f, \eta) \quad \text{and} \quad |f''_{2,\eta}(x)| \leq \frac{5}{\eta^2} \omega_2(f, \eta).$$

PROOF. We can extend f to $\mathbb{R}$ such that $\omega_1(f, \eta, \mathbb{R}) = \omega_1(f, \eta, I)$ and $\omega_2(f, \eta, \mathbb{R}) \leq 5\omega_2(f, \eta, I)$ (see Timan [6, p. 122]). We define for the new function f (differently for $i = 1, 2$, outside of I) the usual Stekelov functions $f_{1,\eta}(x)$ and $f_{2,\eta}(x)$ for all x (2.3)

$$f_{1,\eta}(x) = \frac{1}{h} \int_{-h/2}^{h/2} f(x+u) du, \quad f_{2,\eta}(x) = \frac{1}{h^2} \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} f(x+u+v) du dv$$

and the result follows from [6, p. 163] and the extension theorem mentioned above.

PROOF OF THE THEOREM 2.1. We first deal with the easy implication, that is, $\omega_i(f, h, I) \leq Mh^\alpha$ for $i = 1, 2$ implies

$$|B_n^{(i)}(f; t)| \leq M_1 \left(\min \left\{ n, \sqrt{n/t(1-t)} \right\} \right)^{i-\alpha}.$$

Recalling

$$B_n'(f, t) = n \sum_{k=0}^{n-1} \Delta_{1/n} f \left(\frac{k}{n} \right) P_{k,n-1}(t)$$

and

$$B_n''(f, t) = n(n-1) \sum_{k=0}^{n-2} \Delta_{1/n}^2 f \left(\frac{k}{n} \right) P_{k,n-2}(t),$$

we have $|B_n'(f, t)| \leq n\omega_1(f, 1/n, I) \leq Mn^{1-\alpha}$ and $|B_n''(f, t)| \leq n^2\omega_2(f, 1/n, I) \leq Mn^{2-\alpha}$. Using $|B_n''(f, t)| \leq 5\omega_2(f, \eta, I)[n/t(1-t) + \eta^{-2}]$ (see Becker [2, Lemma 1]) and choosing $\eta = \sqrt{t(1-t)}/n$, we have $|B_n''(f, t)| \leq 10M(n/t(1-t))^{(2-\alpha)/2}$ which completes the estimate of $|B_n''(f, t)|$ (with $M_1 = 10M$). Similarly (using Lemma 2.2), we write

$$\begin{aligned} |B_n'(f, t)| &\leq |B_n'(f - f_{1,\eta}, t)| + |B_n'(f_{1,\eta}, t)| \\ &\leq \|f - f_{1,\eta}\|_{C[0,1]} \frac{1}{t(1-t)} \sum_{k=0}^n |k - nt| P_{k,n}(t) \\ &\quad + \left| n \sum_{k=0}^{n-1} \Delta_{1/n} f_{1,\eta} \left(\frac{k}{n} \right) P_{k,n-1}(t) \right| \\ &\leq \omega_1(f, \eta, I) \left(\frac{n}{t(1-t)} \right)^{1/2} + \sum_{k=0}^{n-1} |f'_{1,\eta}(\xi_k)| P_{k,n-1}(t) \\ &\leq \omega_1(f, \eta, I) \left(\frac{n}{t(1-t)} \right)^{1/2} + \frac{1}{\eta} \omega_1(f, \eta, I), \end{aligned}$$

and choosing $\eta = \sqrt{t(1-t)/n}$, we have $|B'_n(f, t)| \leq 2M(\sqrt{n/t(1-t)})^{1-\alpha}$ ($M_1 = 2M$ for (a)).

To prove the if part (of both (a) and (b)), we now write

$$\begin{aligned} |\Delta_h f(x)| &\leq |\Delta_h \{f(x) - B_n(f, x)\}| + \left| \int_x^{x+h} B'_n(f, u) du \right| \\ &\leq 2 \max_{u=x, x+h} |f(u) - B_n(f, u)| + \int_x^{x+h} |B'_n(f, u)| du, \end{aligned}$$

and similarly,

$$\begin{aligned} |\Delta_h^2 f(x)| &\leq |\Delta_h^2 \{f(x) - B_n(f, x)\}| + |\Delta_h^2 B_n(f, x)| \\ &\leq 4 \max_{u=x, x+h, x+2h} |f(u) - B_n(f, u)| + \left| \int_x^{x+h} (v-x) B''_n(f, v) dv \right| \\ &\quad + \left| \int_{x+h}^{x+2h} (v-x-2h) B''_n(f, v) dv \right|. \end{aligned}$$

Using $f_{i,\eta}$ defined in Lemma 2.2, writing $f = f - f_{i,\eta} + f_{i,\eta}$ and substituting in $|f(u) - B_n(f, u)|$, we obtain $|f(u) - B_n(f, u)| \leq 15\omega_2(f, \sqrt{u(1-u)/n})$ as was done by Strukov and Timan [5], and similarly, $|f(u) - B_n(f, u)| \leq 3\omega_1(f, \sqrt{u(1-u)/n})$. (We can deduce the second inequality with a somewhat worse constant from the first inequality but direct use of the method yields the constant 3.)

Using the estimates above for $\Delta_h f(x)$ and $\Delta_h^2 f(x)$ as well as the estimates for $|f(u) - B_n(f, u)|$, we now obtain

$$|\Delta_h f(x)| \leq 6\omega_1(f, \delta_1(n, x, h)) + \int_x^{x+h} |B'_n(f, u)| du,$$

where $\delta_1(n, x, h) = \max_{u=x, x+h} \sqrt{u(1-u)/n}$, and

$$\begin{aligned} |\Delta_h^2 f(x)| &\leq 4 \cdot 15\omega_2(f, \delta_2(n, x, h)) + \left| \int_x^{x+h} (v-x) B''_n(f, v) dv \right| \\ &\quad + \left| \int_{x+h}^{x+2h} (v-x-2h) B''_n(f, v) dv \right|, \end{aligned}$$

where $\delta_2(n, x, h) = \max_{u=x, x+h, x+2h} \sqrt{u(1-u)/n}$. Without loss of generality we restrict h by $h < \frac{1}{8}$. We estimate $\Delta_h f$ now. Using $|B'_n(f, t)| \leq M_1 n^{1-\alpha}$, we have $\int_x^{x+h} |B'_n(f, u)| du \leq M_1 h n^{1-\alpha}$. Using $|B'_n(f, t)| \leq M_1 (n/t(1-t))^{(1-\alpha)/2}$ in the range $0 \leq x \leq 1-x-h$, we have, for $x \geq h$,

$$\begin{aligned} \int_x^{x+h} |B'_n(f, t)| dt &\leq M_1 h \left(\frac{n}{x(1-x)} \right)^{(1-\alpha)/2} \leq 2M_1 h \left(\frac{n}{2x(1-x)} \right)^{(1-\alpha)/2} \\ &\leq 2M_1 h \left(\frac{n}{(x+h)(1-x-h)} \right)^{(1-\alpha)/2} \end{aligned}$$

and, for $x < h$,

$$\begin{aligned} \int_x^{x+h} |B'_n(f, t)| dt &\leq M_1 \int_0^{x+h} \left(\frac{n}{t(1-t)} \right)^{(1-\alpha)/2} dt \\ &< \frac{4}{3} M_1 n^{(1-\alpha)/2} \frac{2}{(\alpha+1)} (x+h)^{(\alpha+1)/2} \\ &\leq \frac{8}{3} M_1 n^{(1-\alpha)/2} (2h) \left(\frac{1}{(x+h)(1-x-h)} \right)^{(1-\alpha)/2} \\ &\leq \frac{16}{3} M_1 h \left(\frac{n}{(x+h)(1-x-h)} \right)^{(1-\alpha)/2}. \end{aligned}$$

For $1 - x \leq x + h$ ($1 - x - h \leq x$) the roles of x and $1 - x$ and that of x and $x + h$ in the proof are reversed and together with the above we have

$$\int_x^{x+h} |B'_n(f, u)| du \leq 6M_1 h \left(\min \left\{ n, \sqrt{n/x(1-x)}, \sqrt{n/(x+h)(1-x-h)} \right\} \right)^{1-\alpha}.$$

We can now choose n big enough (for the given x and h) such that, $\delta_1(n, x, h) \leq h/T_1$ while $\min \{ n, \sqrt{n/x(1-x)}, \sqrt{n/(x+h)(1-x-h)} \} \leq 2T_1/h$ for $T_1 > 1$ which will satisfy $T_1^\beta > 12$. This is possible since for $x(1-x) \leq (x+h)(1-x-h)$ we choose the smallest $n, n \geq 1$, for which

$$\delta(n, x, h) = \sqrt{\frac{(x+h)/1-x-h}{n}} < \frac{h}{T_1};$$

if $n = 1$ we use $1 \leq 2T_1/h$, otherwise

$$\begin{aligned} \min \left\{ n, \sqrt{n/x(1-x)}, \sqrt{n/(x+h)(1-x-h)} \right\} \\ \leq \sqrt{n/(x+h)(1-x-h)} \\ \leq \sqrt{2} \sqrt{(n-1)/(x+h)(1-x-h)} \leq \sqrt{2} T_1/h \leq 2T_1/h. \end{aligned}$$

For $x(1-x) \geq (x+h)(1-x-h)$ take x instead of $x+h$, and $1-x$ instead of x .

Combining the estimate of $\Delta_h f(x)$ and the choice of n , we have

$$\begin{aligned} (2.4) \quad |\Delta_h f(x)| &\leq 6\omega_1(f, h/T_1) + 6M_1 h (2T_1/h)^{1-\alpha} \\ &\equiv A_1 \omega_1(f, h/T_1) + B_1 h (T_1/h)^{1-\alpha}. \end{aligned}$$

Note now that in (2.4) the right side does not depend on x nor on $B_n(f, t)$.

We now proceed with the estimate of $\Delta_h^2 f(x)$ for $0 < h < \frac{1}{8}$. Using $|B''_n(f, t)| \leq M_1 n^{2-\alpha}$, we have

$$\int_x^{x+h} (v-x) |B''_n(f, v)| dv \leq M_1 h^2 n^{2-\alpha} / 2$$

and

$$\int_{x+h}^{x+2h} (v-x-2h) |B''_n(f, v)| dv \leq M_1 h^2 n^{2-\alpha} / 2.$$

For $x \leq 1 - x - 2h$, and therefore

$$(x+2h)(1-x+2h) \geq x(1-x)$$

(but $(x+h)(1-x-h)$ may be bigger than either), we have, for $x \geq h$,

$$\begin{aligned} \int_x^{x+h} (v-x) |B''_n(f, v)| dv &\leq M_1 h^2 \left(\frac{n}{x(1-x)} \right)^{1-\alpha/2} \frac{1}{2} \leq \frac{3}{2} M_1 h^2 \left(\frac{n}{3x(1-x)} \right)^{1-\alpha/2} \\ &\leq M_1 \frac{3}{2} h^2 \left(\frac{n}{(x+2h)(1-x-2h)} \right)^{1-\alpha/2} \end{aligned}$$

and similarly

$$M_1 h^2 \left(\frac{n}{x(1-x)} \right)^{1-\alpha/2} \frac{1}{2} \leq M_1 h^2 \left(\frac{n}{(x+h)(1-x-h)} \right)^{1-\alpha/2}.$$

We also have (still for $x \leq 1 - x - 2h$), for $x \leq h$,

$$\begin{aligned} \int_x^{x+h} (v-x) |B_n''(f, v)| dv &\leq M_1 n^{1-\alpha/2} \int_x^{x+h} (v-x) (v(1-v))^{-1+\alpha/2} dv \\ &\leq M_1 n^{1-\alpha/2} \frac{4}{3} \int_0^{x+h} v^{\alpha/2} dv = M_1 \frac{4}{3} n^{1-\alpha/2} \frac{1}{1+\alpha/2} (x+h)^{1+\alpha/2} \\ &\leq M_1 \frac{4}{3} n^{1-\alpha/2} (x+2h)^{1+\alpha/2} \leq M_1 \frac{4}{3} n^{1-\alpha/2} (3h)^2 (x+2h)^{-1+\alpha/2} \\ &\leq 12 M_1 h^2 \left(\frac{n}{(x+2h)(1-x-2h)} \right)^{1-\alpha/2}. \end{aligned}$$

The estimate of $|\int_{x+h}^{x+2h} (v-x-2h) |B_n''(f, v)| dv|$ for $x \leq 1 - x - 2h$ and the estimate of both terms for $x \geq 1 - x - 2h$ are similar. (In the latter the roll of x and $x+2h$ and that of x and $1-x$ are reversed.) We now have

$$\begin{aligned} |\Delta_h^2 f(x)| &\leq 60 \omega_2(f, \delta_2(n, x, h)) \\ &\quad + M_1 h^2 24 \left(\min(n, \sqrt{n/x(1-x)}, \sqrt{n/(x+h)(1-x-h)}, \right. \\ &\quad \left. \sqrt{n/(x+2h)(1-x-2h)}) \right)^{2-\alpha}. \end{aligned}$$

Choosing n (in a similar way to the choice made to prove (2.4)) such that $\delta_2(n, x, h) \leq h/T_2$ while

$$\begin{aligned} \min(n, \sqrt{n/x(1-x)}, \sqrt{n/(x+h)(1-x-h)}, \\ \sqrt{n/(x+2h)(1-x-2h)}) \leq 2T_2/h \end{aligned}$$

for some $T_2, T_2 > 1$ (we will choose T_2 such that $T_2^\beta > 120$), we have

$$(2.5) \quad |\Delta_h^2 f(x)| \leq A_2 \omega_2(f, h/T_2) + h^2 B_2 (T_2/h)^{2-\alpha},$$

where $A_2 = 60$ and $B_2 = 48M_1$.

We will now show that $|\Delta_h f(x)| \leq Kh^\beta$ for some β implies $|\Delta_h f(x)| \leq M_2 h^\alpha$ and $|\Delta_h^2 f(x)| \leq M_2 h^\alpha$ for (a) and (b) of our theorem. Choosing T_1 such that $T_1^\beta > 2A_1 = 12$ we write

$$\begin{aligned} |\Delta_h f(x)| &\leq A_1 \omega_1(f, h/T_1) + B_1 h (T_1/h)^{1-\alpha} \\ &\leq A_1^2 \omega_1(f, h/T_1^2) + B_1 T_1^{1-\alpha} h^\alpha (1 + A_1 T_1^{-\alpha}) \\ &\leq A_1^s \omega_1(f, h/T_1^s) + B_1 T_1^{1-\alpha} h^\alpha (1 + A_1 T_1^{-\alpha} + \dots + A_1^{s-1} T_1^{-(s-1)\alpha}) \\ &\leq A_1^s \omega_1(f, h/T_1^s) + B_1 T_1^{1-\alpha} h^\alpha (1 - A_1 T_1^{-\alpha})^{-1} \end{aligned}$$

and since, for $T_1^\beta > 2A_1$, $A_1^s \omega_1(f, h/T_1^s)$ tends to zero, we have $|\Delta_h f(x)| \leq B_1 T_1^{1-\alpha} (1 - A_1 T_1^{-\alpha})^{-1} h^\alpha$ which, since $T_1^\beta > T_1^\beta > 2A_1$, implies $|\Delta_h f(x)| \leq M_2 h^\alpha$. Similarly, for $T_2^\beta > 2A_2$ we have

$$\begin{aligned} |\Delta_h^2 f(x)| &\leq A_2 \omega_2(f, h/T_2) + B_2 h^2 (T_2/h)^{2-\alpha} \\ &\leq A_2^s \omega_2(f, h/T_2^s) + B_2 T_2^{2-\alpha} (1 - A_2 T_2^{-\alpha})^{-1} h^\alpha \end{aligned}$$

and hence $|\Delta_h^2 f(x)| \leq B_2 T_2^{2-\alpha} (1 - A_2 T_2^{-\alpha})^{-1} h^\alpha \leq M_2 h^\alpha$.

3. Remarks and conclusions. For proving that $f(x)$ belongs to Lipschitz class α we need the a priori condition that $f(x)$ satisfies Lipschitz condition of order β (no matter how small). This a priori condition is not desirable and it seems interesting to find whether it can be dropped. For the case below it will be shown that it can be.

THEOREM 3.1. For $f \in C[0, 1]$, $|B'_n(f, x)| \leq M_1 \{\min(n^2, n/x(1-x))\}^{(1-\alpha)/2}$ and f monotonic imply $|\Delta_h f(x)| \leq Mh^\alpha$.

PROOF. For f monotonic $\Delta_{1/n} f(k/n)$ is of fixed sign, and therefore,

$$|B'_n(f, x)| = n \sum_{k=0}^{n-1} \left| \Delta_{1/n} f\left(\frac{k}{n}\right) \right| P_{k,n-1}(x) \leq M_1 \{\min(n^2, n/x(1-x))\}^{(1-\alpha)/2}.$$

Using Stirling's estimate for $m!$, we obtain, for $0 < k < n-1$,

$$P_{k,n-1}\left(\frac{k}{n-1}\right) \geq C \frac{1}{\sqrt{n}((k/(n-1))(1-k/(n-1)))^{1/2}}$$

for some $C > 0$, independent of n and k , and therefore, for those k using $x = k/(n-1)$,

$$\left| B'_n\left(f, \frac{k}{n-1}\right) \right| \geq n \left| \Delta_{1/n} f\left(\frac{k}{n}\right) \right| P_{k,n-1}\left(\frac{k}{n-1}\right)$$

and therefore

$$\left| \Delta_{1/n} f\left(\frac{k}{n}\right) \right| \leq \frac{M_1}{C} \left(\frac{(k/(n-1))(1-k/(n-1))}{n} \right)^{\alpha/2} \leq \frac{M_1}{C} n^{-\alpha/2}.$$

For $x = 0$ and $x = 1$ we have $|\Delta_{1/n} f(0)| \leq M_1 n^{-\alpha}$ and $|\Delta_{1/n} f((n-1)/n)| \leq M n^{-\alpha}$.

Therefore Theorem 2.1 implies that $f \in \text{Lip } \alpha$ using $|\Delta_{1/n} f(k/n)| \leq K n^{-\alpha/2}$ for all k and n , or $f \in \text{Lip } (\alpha/2)$ ($\beta = \alpha/2$).

A more interesting application of Theorem 2.1 is the estimate for the Kantorovich modification of Bernstein polynomials. The Kantorovich operator is given by

$$(3.1) \quad K_n(f, t) = \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} (n+1) \int_{k/n+1}^{(k+1)/n+1} f(u) du \\ = \frac{d}{dt} B_{n+1}(F, t),$$

where $F(x) = \int_0^x f(u) du$.

THEOREM 3.2. For $f \in C[0, 1]$ and $\alpha \leq 1$ we have $|\Delta_h f(x)| \leq Mh^\alpha$ if and only if $|K'_n(f, t)| \leq M_1 (\min(n, \sqrt{n/x(1-x)}))^{1-\alpha}$.

REMARK. We observe that here a condition $f \in \text{Lip } \beta$ is not required.

PROOF. We observe $|\Delta_h f| \leq Mh^\alpha$ implies $|\Delta_h^2 F(x)| \leq Mh^{1+\alpha}$ and therefore

$$|B''_{n+1}(F, x)| = |K'_n(f, x)| \leq M_1 \left(\min(n+1, \sqrt{n+1/x(1-x)}) \right)^{1-\alpha}.$$

Assume now that

$$|K'_n(f, x)| = |B''_{n+1}(F, x)| \leq M \left(\min(n, \sqrt{n/x(1-x)}) \right)^{2-(1+\alpha)};$$

this implies for $F(x) \in \text{Lip } 1$, which follows $|\Delta_h F(x)| \leq h\|f\|$, that $|\Delta_h^2 F(x)| \leq M_1 h^{1+\alpha}$ and this implies $|\Delta_h f(x)| \leq M_1 h^\alpha$.

Actually the following even "more general" statement is valid.

THEOREM 3.3. *For $f \in L_p[0, 1]$, $p > 1$, and $\alpha \leq 1$ the condition $|\Delta_h f(x)| \leq Mh^\alpha$ is equivalent to $|K'_n(f, t)| \leq M_1(\min(n, \sqrt{n/x(1-x)}))^{1-\alpha}$.*

Note that though the initial condition is $f \in L_p[0, 1]$, we show that f is a continuous function. The proof follows the observation that $F(x) = \int_0^x f(u) du$, and therefore

$$\begin{aligned} |F(x+h) - F(x)| &= \left| \int_x^{x+h} f(u) du \right| \leq \left\{ \int_x^{x+h} 1 \right\}^{1/q} \left\{ \int_x^{x+h} |f(u)|^p du \right\}^{1/p} \\ &\leq h^{1/q} \|f\|_{L_p}, \end{aligned}$$

where $1/q + 1/p = 1$. Now $F(x) \in \text{Lip } q^{-1}$ and satisfies the condition on f in Theorem 2.1 which completes the proof, recalling $K'_n(f, t) = B''_n(F, t)$. (Of course we have just shown that f is equivalent to a function in $C[0, 1]$ satisfying $\text{Lip}^* \alpha$ condition.)

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