MARKOV'S INEQUALITY FOR POLYNOMIALS WITH REAL ZEROS

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ABSTRACT. Markov's inequality asserts that $\|p'_n\| \leq n^2 \|p_n\|$ for any polynomial p_n of degree n . (We denote the supremum norm on $[-1, 1]$ by $\|\cdot\|$.) In the case that p_n has all real roots, none of which lie in $[-1, 1]$, Erdős has shown that $\|p'_n\| \leq en \|p_n\|/2$. We show that if p_n has $n - k$ real roots, none of which lie in $[-1, 1]$, then $\|p'_n\| \leq cn(k + 1) \|p_n\|$, where c is independent of n and k . This extension of Markov's and Erdős' inequalities was conjectured by Szabados.

Introduction. Markov's inequality asserts that

$$(1) \quad \|p'_n\|_{[-1,1]} \leq n^2 \|p_n\|_{[-1,1]}$$

for any polynomial $p_n \in \pi_n$ [2 and 3]. (π_n denotes the algebraic polynomials of degree at most n and $\|\cdot\|_A$ denotes the supremum norm on A .) Erdős [1] in 1940 offered the following refinement of Markov's inequality. If $p_n \in \pi_n$ and p_n has all its roots in $\mathbf{R} - (-1, 1)$, then

$$(2) \quad \|p'_n\|_{[-1,1]} \leq \frac{en}{2} \|p_n\|_{[-1,1]}.$$

Inequality (1) iterates to give bounds for the k th derivative of a polynomial. However, we cannot proceed inductively with inequality (2) since some of the roots of the derivatives may be in $[-1, 1]$. With this in mind, Szabados and Varma established a version of (2) for polynomials of degree n with all real roots and at most one root in $[-1, 1]$, namely, for such a polynomial p_n ,

$$(3) \quad \|p'_n\|_{[-1,1]} \leq c_1 n \|p_n\|_{[-1,1]},$$

where c_1 is independent of n [5]. This, of course, yields the following inequality:

$$(4) \quad \|p''_n\|_{[-1,1]} \leq c_2 n^2 \|p_n\|_{[-1,1]}$$

for any $p_n \in \pi_n$ that has all its roots in $\mathbf{R} - (-1, 1)$. In [6] Szabados proposed the following

Conjecture. If p_n is a polynomial of degree n and p_n has at least $n - k$ roots in $\mathbf{R} - (-1, 1)$, then there is a constant c ($c \leq 9$) so that

$$(5) \quad \|p'_n\|_{[-1,1]} \leq cn(k + 1) \|p_n\|_{[-1,1]}.$$

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It is our intention to prove this slightly strengthened form of Szabados' conjecture. In its original form the conjecture had the additional assumption that all the roots of p_n be real. Up to the constant this result is best possible; Szabados in [6] constructs polynomials p_n of degree n with $n - k$ roots in $\mathbf{R} - (-1, 1)$ so that

$$\|p'_n\|_{[-1,1]} \geq \frac{n \cdot k}{2} \|p_n\|_{[-1,1]} \quad (0 < k \leq n).$$

It is apparent from (1) and (2) that the best constant must depend on k . Some related results may be found in [4].

Inequalities for the higher derivatives of polynomials with real roots can now be derived straightforwardly from (5). For example,

THEOREM. *If $p_n \in \pi_n$ has at least $n - k$ zeros in $\mathbf{R} - (-1, 1)$, then*

$$\|p_n^{(m)}\|_{[-1,1]} \leq c_m \frac{n!(k+m)!}{(n-m)!k!} \|p_n\|_{[-1,1]},$$

where $c_m \leq 9^m$ depends only on m .

2. Proof of the Conjecture. Let c_{2k} be the $2k$ th Chebychev polynomial shifted to the interval $[0, 2]$ and normalized to have lead coefficient 1. Let $\alpha_1 < \alpha_2 < \dots < \alpha_k$ be the roots of c_{2k} in $[1, 2]$ and let

$$(6) \quad t_k := \prod_{i=1}^k (x - \alpha_i).$$

LEMMA 1. *The polynomial $q_k := (x + 2m/k)^{m+k} t_k(x)$ has the following property. If $\alpha_0 = 0$ and $\alpha_{n+1} = 1$, then, for $i = 1, 2, \dots, n$,*

$$(7) \quad \left\| \left(x + \frac{2m}{k} \right)^{m+k} t_k \right\|_{[\alpha_{i-1}, \alpha_i]} > \left\| \left(x + \frac{2m}{k} \right)^{m+k} t_k \right\|_{[\alpha_i, \alpha_{i+1}]},$$

where the maximums on successive intervals occur with alternating sign.

PROOF. Let $0 < \beta_1 < \dots < \beta_k$ be the roots of c_{2k} in $[0, 1]$. We observe that $x/(x - \beta_i)$ is positive and decreasing on $(\beta_i, \infty]$ and that c_{2k} equioscillates on the intervals in question (i.e. c_{2k} satisfies (7) with equality). We now note that

$$x^k t_k = \left(\prod_{i=1}^k \frac{x}{x - \beta_i} \right) c_{2k}$$

satisfies the conclusion of Lemma 1. To finish the proof we need only observe that $(x + 2m/k)^{m+k}/x^k$ is decreasing on $[0, 2]$. $\square$

Let $n = 2k + m$ and let

$$s_n(x) := \frac{1}{(1 + m/k)^n} q_k \left(\left(1 + \frac{m}{k} \right) x + \left(1 - \frac{m}{k} \right) \right).$$

(We have shifted from $[-2m/k, 2]$ to $[-1, 1]$.) This polynomial will act as a kind of near extremal polynomial for the Conjecture. Let $\gamma_1 < \gamma_2 < \dots < \gamma_k$ be the roots of s_n in $(-1, 1)$. We collect the properties of s_n that we require in the next lemma.

LEMMA 2. For $n = m + 2k$ and s_n as above:

- (a) $s_n(x) = (x + 1)^{m+k} \prod_{i=1}^k (x - \gamma_i)$,
- (b) $\sum_{i=1}^k (1/(1 - \gamma_i)) \leq 4k(n - k)$, and
- (c) for $i = 1, \dots, k$, $\gamma_0 = -1$ and $\gamma_{k+1} = 1$

$$\|s_n\|_{[\gamma_{i-1}, \gamma_i]} \geq \|s_n\|_{[\gamma_i, \gamma_{i+1}]}$$

PROOF. Parts (a) and (c) are immediate from the construction of s_n . Part (b) follows from the observation that, for α_i as in (6).

$$\sum_{i=1}^k \frac{1}{2 - \alpha_i} \leq \frac{c'_{2k}(2)}{c_{2k}(2)} = 4k^2$$

and the observation that

$$1 - \gamma_i = \frac{1}{1 + m/k} (2 - \alpha_i). \quad \square$$

Let $p_n^* \in \pi_n$ maximize

$$(8) \quad |p_n'(1)| / \|p_n\|_{[-1, 1]},$$

where the maximum is taken over all polynomials in π_n that have all but at most k roots in $\mathbf{R} - (-1, 1)$. The information we need about p_n^* is contained in the next lemma.

LEMMA 3. Let p_n^* be as above. Then

- (a) p_n^* has k simple roots $\delta_1 < \dots < \delta_k$ in $(-1, 1)$, p_n^* has $n - k$ roots at ± 1 , and p_n^* achieves its maximum modulus on each of the intervals $[-1, \delta_1]$, $[\delta_1, \delta_2]$, $\dots$, $[\delta_k, 1]$.
- (b) Either p_n^* has no roots in $[-1, \infty)$ or p_n^* has exactly one root at 1.

PROOF. The proof of (a) is a simple and standard perturbation argument (if p_n^* did not satisfy (a) then it would be possible to perturb p_n^* to reduce its norm on $[-1, 1]$ without decreasing the derivative at 1). We will prove only that p_n^* has no roots in $(1, \infty)$, the other parts are similar. First suppose that p_n^* has two roots at $\alpha > 1$ and $\beta > 1$. Consider

$$v_n(x) := \frac{p_n^*(x)(x - 1)^2}{(x - \alpha)(x - \beta)}.$$

Then for sufficiently small $\varepsilon > 0$:

- (i) $\|p_n^* - \varepsilon v_n\|_{[-1, 1]} < \|p_n^*\|_{[-1, 1]}$,
- (ii) $|(p_n^* - \varepsilon v_n)'(1)| = |p_n^{*'}(1)|$,
- (iii) $p_n^* - \varepsilon v_n$ has all but at most k roots in $\mathbf{R} - (1, 1)$.

Part (iii) follows since $(x - \alpha)(x - \beta) - \varepsilon(x - 1)^2$ has two roots in $[1, \infty]$ for sufficiently small ε . (Note α may equal β .) However, this contradicts the maximality of p_n^* .

Next we suppose that p_n^* has exactly one (nonrepeat) root at $\alpha > 1$. Now we argue as before by considering

$$v_n(x) := p_n^*(x) \frac{(x - 1)^2}{(x - 1)(x - \alpha)}.$$

If $p_n^*(1) \neq 0$ we must observe that in this case $\text{sign}(p_n^*(1)) = \text{sign}(p_n^{**}(1))$ and, hence, that

$$v_n'(1) = \frac{(1 - \alpha)p_n^*(1)}{(1 - \alpha)^2}$$

has the opposite sign to $p_n^{**}(1)$. The last observation requires noticing that if $p_n^{**}(1)$ has opposite sign to $p_n^*(1)$, then p_n^{**} has all its zeros in $(-\infty, 1]$ and, hence, $p_n^{**}(1) \neq 0$. Thus, $|p_n^{**}|$ is increasing on $[1, \infty)$, $|p_n^*|$ is decreasing on $[1, \alpha)$ and $p_n^*(x + (\alpha - 1))$ violates the maximality assumptions on $[-1, 1]$.

Part (b) follows since if p_n^* has two or more zeros at 1, then $p_n'(1)$ would also equal zero. $\square$

LEMMA 4. *If $p_n \in \pi_n$ has at least $(n - k)$ roots in $\mathbf{R} - (-1, 1)$, then*

$$|p_n'(1)| \leq \frac{9}{2}(k + 1)n\|p_n\|_{[-1, 1]}.$$

PROOF. If $2k \geq n$, then the lemma follows from Markov's inequality, so we may suppose $2k < n$. Suppose there exists p_n , as above, so that

$$|p_n'(1)| > \frac{9}{2}(k + 1)(n - k)\|p_n\|_{[-1, 1]},$$

and let q_n be the maximal such p_n . By Lemma 3, this q_n equioscillates $k + 1$ times on $[-1, 1]$.

We shall first consider the case where q_n has no root at 1.

The key to the proof is to observe that the roots of q_n lie to the left of the roots of s_n (as defined in Lemma 3). We may write

$$q_n(x) = (x + 1)^{n-k} \prod_{i=1}^k (x - \rho_i),$$

where $-1 < \rho_1 < \dots < \rho_k < 1$. Also,

$$s_n(x) = (x + 1)^{n-k} \prod_{i=1}^k (x - \gamma_i).$$

The claim is that $\gamma_i \geq \rho_i$ for each i . This is seen as follows. Choose the largest i for which $\rho_i > \gamma_i$. Then pick η so that $\|\eta q_n\|_{[\gamma_i, 1]} = \|s_n\|_{[\gamma_i, 1]}$. (We will specify the sign of η later.) We can deduce from the equioscillation of q_n that $\eta q_n - s_n$ has at least $k - i$ roots on $[\beta, 1]$, where β is the first point greater than ρ_i where ηq_n achieves its maximum modulus. From Lemma 2(c) we deduce that $\eta q_n - s_n$ has at least $i - 1$ roots on $(-1, \alpha)$, where α is the largest point less than γ_i where s_n achieves its maximum modulus. We need only observe that if we choose the sign of η so that

$$\text{sign } \eta q_n(\beta) = -\text{sign } s_n(\alpha),$$

then $\eta q_n - s_n$ must have 2 roots in (α, β) . Thus, $\eta q_n - s_n$ has $n + 1$ roots which is a contradiction and we conclude that $\rho_i \leq \gamma_i$.

We now observe that, since $\rho_i \leq \gamma_i < 1$,

$$\begin{aligned} \frac{|q'_n(1)|}{\|q_n\|_{[-1,1]}} &= \frac{q'_n(1)}{q_n(1)} = \sum_{i=1}^k \frac{1}{1-\rho_i} + \sum_{i=1}^{n-k} \frac{1}{1-(-1)} \\ &\leq \sum_{i=1}^k \frac{1}{1-\gamma_i} + \frac{n-k}{2} \leq 4k(n-k) + \frac{n-k}{2}, \end{aligned}$$

where the later inequality follows from Lemma 2. This is a contradiction.

In the case where q_n has exactly one root at 1 we proceed as follows. Let $d > 1$ be the unique point in $(1, \infty)$, where $|q_n(d)| = \|q_n\|_{[-1,1]}$. We can now consider q_n on $[-1, d]$. We note that

$$\frac{|q'_n(d)|}{\|q_n\|_{[-1,1]}} \geq \frac{|q'_n(1)|}{\|q_n\|_{[-1,1]}}$$

since $|q'_n|$ is increasing on $[1, \infty)$. We can repeat verbatim the argument of the first part applied to

$$\tilde{q}(x) = q_n\left(x\left(\frac{d+1}{2}\right) + \left(\frac{d-1}{2}\right)\right)$$

with k replaced by $k+1$. This allows us to deduce the contradiction that

$$\frac{|q'(1)|}{\|q_n\|_{[-1,1]}} \leq \frac{|\tilde{q}_n(1)|}{\|\tilde{q}_n\|_{[-1,1]}} \leq 4(k+1)(n-k) + \frac{n-k}{2}. \quad \square$$

The proof of the Conjecture is now straightforward.

PROOF OF CONJECTURE. Let p_n be a polynomial of degree n with $n-k$ roots in $\mathbf{R} - (-1, 1)$. Let x_0 be a point in $[-1, 1]$, where p'_n achieves its maximum modulus. We suppose $x_0 \leq 0$ ($x_0 > 0$ follows analogously). Let $ax + b$ map $[x_0, 1]$ one-to-one onto $[-1, 1]$ in such a way that $x_0 \rightarrow 1$.

Note that $|a| < 2$. Thus, if $v_n(ax + b) = p_n(x)$, then

$$\frac{|p'_n(x_0)|}{\|p_n\|_{[-1,1]}} \leq \frac{2|v'_n(1)|}{\|v_n\|_{[-1,1]}} = 9n(k+1),$$

where the last inequality follows from Lemma 4 and the observation that v_n has at least as many roots as p_n in $\mathbf{R} - [-1, 1]$. $\square$

REFERENCES

1. P. Erdős, *On extremal properties of the derivatives of polynomials*, Ann. of Math. (2) **41** (1940), 310–313.
2. G. G. Lorentz, *Approximation of functions*, Holt, Reinhart and Winston, New York, 1966.
3. A. A. Markov, *On a problem of D. I. Mendeleeu*, Acad. Sci., St. Petersburg **62** (1889), 1–24.
4. A. Máté, *Inequalities for derivatives of polynomials with restricted zeros*, Proc. Amer. Math. Soc. **88** (1981), 221–225.
5. J. Szabados and A. K. Varma, *Inequalities for derivatives of polynomials having real zeros*, Approximations Theory. III (E. W. Cheney, ed.), Academic Press, New York, 1980, pp. 881 – 888.
6. J. Szabados, *Bernstein and Markov type estimates for the derivative of a polynomial with real zeros*, Functional Analysis and Approximation (P. L. Butzer, B. Sz.-Nagy and E. Görlick, eds.), Birkhäuser, Basel, 1981, pp. 177–188.

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