PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 93, Number 1, January 1985

A NOTE ON STRONG EXTREME AND STRONGLY EXPOSED POINTS IN BOCHNER L^p-SPACES

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ABSTRACT. A strongly exposed (strongly extreme) vector-valued L^p-function takes only strongly exposed (strongly extreme) values almost everywhere on its support. We remove the restrictions, concerning the range space and the measure, in previous papers of the author and M. A. Smith.

It has been shown in [2] that strongly exposed points of balls in Bochner L^p-spaces $L^p(\mu, V)$ have to take strongly exposed values, provided that μ is a *Radon* measure on a locally compact space or, for general measures, that V is *separable*. M. A. Smith [5] verified the same statement for strong extreme points (i.e., those points x for which only null sequences (z_n) can satisfy $\|x \pm z_n\| \overset{n}{\to} \|x\|$). The aim of this note is to show that in both cases the restrictions concerning the measure and the range space are superfluous. In fact, we have the following theorems.

THEOREM 1. *Let* (Ω, Σ, μ) *be a positive measure space*, V *a Banach space*, $1 < p < \infty$, *and* $x \in L^p(\mu, V)$. *Then* x *is strongly extreme if and only if for almost all* $t \in \Omega$, $x(t)$ *is strongly extreme.*

THEOREM 2. *Let* $x \neq 0$ *be as above and* $g \in L^p(\mu, V)'$ $(\cong L^q_*(\mu, V'))$. *Then* g *strongly exposes* x *if and only if* $|g|$ *strongly exposes* $|x|$, *and for almost all* $t \in \Omega$, $g(t)$ *strongly exposes* $x(t)$ *or* $g(t) = 0 = x(t)$.

Here $L^q_*(\mu, V')$ denotes the Banach space of all weak*-measurable L^q-functions modulo weak*-equivalence (see [3, pp. 80, 97]), and, as in [2], $|x|$ and $|g|$ denote the moduli $t \mapsto \|x(t)\|$ and $\|g(t)\|$, respectively, and q is the index conjugate to p.

PROOF OF THEOREM 2. As observed in [2] we may assume w.l.o.g. that μ is finite and complete. Let K be the Stonean space of μ's measure algebra, and set $m(C) := \mu(M)$, where $C \subset K$ is the clopen set corresponding to $M \in \Sigma$. It is well known that m extends to a regular Borel measure on K, vanishing on nowhere dense sets. Furthermore, every lifting ρ of $L^\infty(\mu)$ induces an inverse measure preserving map $\hat{\rho}\colon \Omega \to K$ (see [4, pp. 425, 435] for details). Clearly the composition with $\hat{\rho}$ defines isometric isomorphisms $L^p(m, V) \cong L^p(\mu, V)$ and $L^q_*(m, V') \cong L^q_*(\mu, V')$ such that

$$\langle y, f \rangle = \langle y \circ \hat{\rho}, f \circ \hat{\rho} \rangle \quad \text{for all } y \in L^p(m, V), f \in L^q_*(m, V').$$

Received by the editors January 3, 1984.
1980 *Mathematics Subject Classification*. Primary 46E40; Secondary 46B20.
Key words and phrases. Bocher L^p-space, hyperstonean, Stonean transform, strong extreme point, strongly exposed point.

Thus $f \circ \hat{\rho}$ in $L^q_*(\mu, V')$ strongly exposes $y \circ \hat{\rho}$ in $L^p(\mu, V)$ if and only if f strongly exposes y. An appeal to [**2**, Theorem 2] yields the desired result. □

PROOF OF THEOREM 1. Smith has shown that for a function x in $L^p(\mu, V)$ to be strongly extreme it is sufficient that almost all of its values are strongly extreme, and that this condition is necessary provided that μ is Radon or V is separable. (Observe that in our definition the zero vector is strongly extreme.) In order to verify the necessity in the general case, we assume w.l.o.g. that μ is complete and finite—if x is strongly extreme, then also $\chi_M x$ is $(M \in \Sigma)$—and then proceed as in Theorem 2. □

The following improvement of Theorem 3′ in [**2**] is a trivial corollary.

COROLLARY. *For each strongly exposed* $x \in L^p(\mu, V)$ *almost all values* $x(t)$ *are strongly exposed or zero.*

REMARKS. 1. Apart from the case that V is smooth [**2**, Theorem 4], we know nothing about the converse of the corollary, but it seems that one should try measurable selection theorems. Using the Stonean representation $L^p(m, V)$ of $L^p(\mu, V)$ and looking at suitable partitions of K, the general case is easily reduced to: x separably valued, norm continuous (compare [**4**, Theorem 2.2.b]), $|x| = 1$, and each weak*-measurable bounded function is (weak*-) equivalent to a weak*-continuous one. This suggests that even continuous selection theorems might work.

2. Theorem 1 becomes false if we replace "strongly extreme" by "extreme". In [**1**] a counterexample is given where μ is Lebesgue measure on $[0,1]$. In a forthcoming paper [**6**] it is shown that essentially the same counterexample works for *all* measures μ.

3. In [**1**, Remark 4.b] it has been asked whether for the Stonean representation $T\colon L^p(\mu, V) \to L^p(m, V)$ it is true that x and Tx have essentially the same values, i.e., there are null sets $N \subset \Omega$ and $D \subset K$ such that $x(\Omega \setminus N) = Tx(K \setminus D)$. With $\hat{\rho}$ in mind one is tempted to say "yes", but that idea breaks down when it comes to showing that the natural candidate for $K \setminus D$, namely $\hat{\rho}(\Omega)$, is a co-null set. Although it is dense in K, it is not m-measurable in general. S. Graf and, independently, A. Iwanik have communicated an example of two isomorphic measure algebras and a scalar-valued L^∞-function x whose image under the isometry induced by the Boolean isomorphism has range even disjoint to that of x.

BIBLIOGRAPHY

1. P. Greim, *An extremal vector-valued L^p-function taking no extremal vectors as values*, Proc. Amer. Math. Soc. **84** (1982), 65–68.
2. ______, *Strongly exposed points in Bochner L^p-spaces*, Proc. Amer. Math. Soc. **88** (1983), 81–84.
3. A. Ionescu-Tulcea and C. Ionescu-Tulcea, *Topics in the theory of lifting*, Springer-Verlag, Berlin, 1969.
4. D. Sentilles, *Decomposition of weakly measurable functions*, Indiana Univ. Math. J. **32** (1983), 425–437.
5. M. A. Smith, *Strongly extreme points in $L^p(\mu, X)$*, Rocky Mountain J. Math. (to appear).
6. P. Greim, *An extension of J. A. Johnson's characterization of extremal vector-valued L^p-functions* (submitted).

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